Effective Field Theory for Positronium in Relativistic Motion

Jinchen He
(Supervised by Prof. Xiangdong Ji)

Nuclear Theory Seminar

2025/04

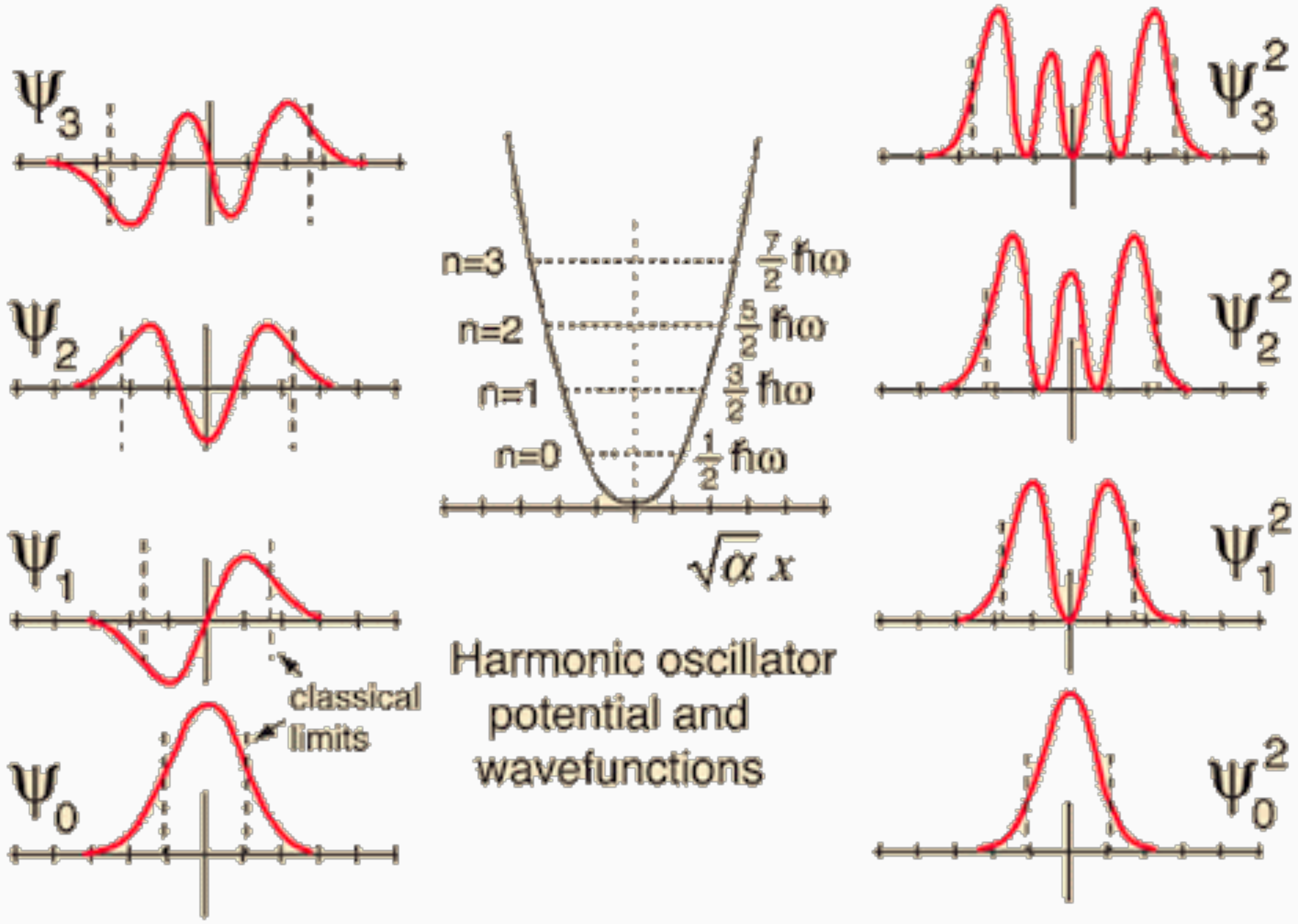
UNIVERSITY OF MARYLAND

Argonne NATIONAL LABORATORY

Contents

Description of Bound States

A Question Since My Undergrad QFT Course



©C.R. Nave

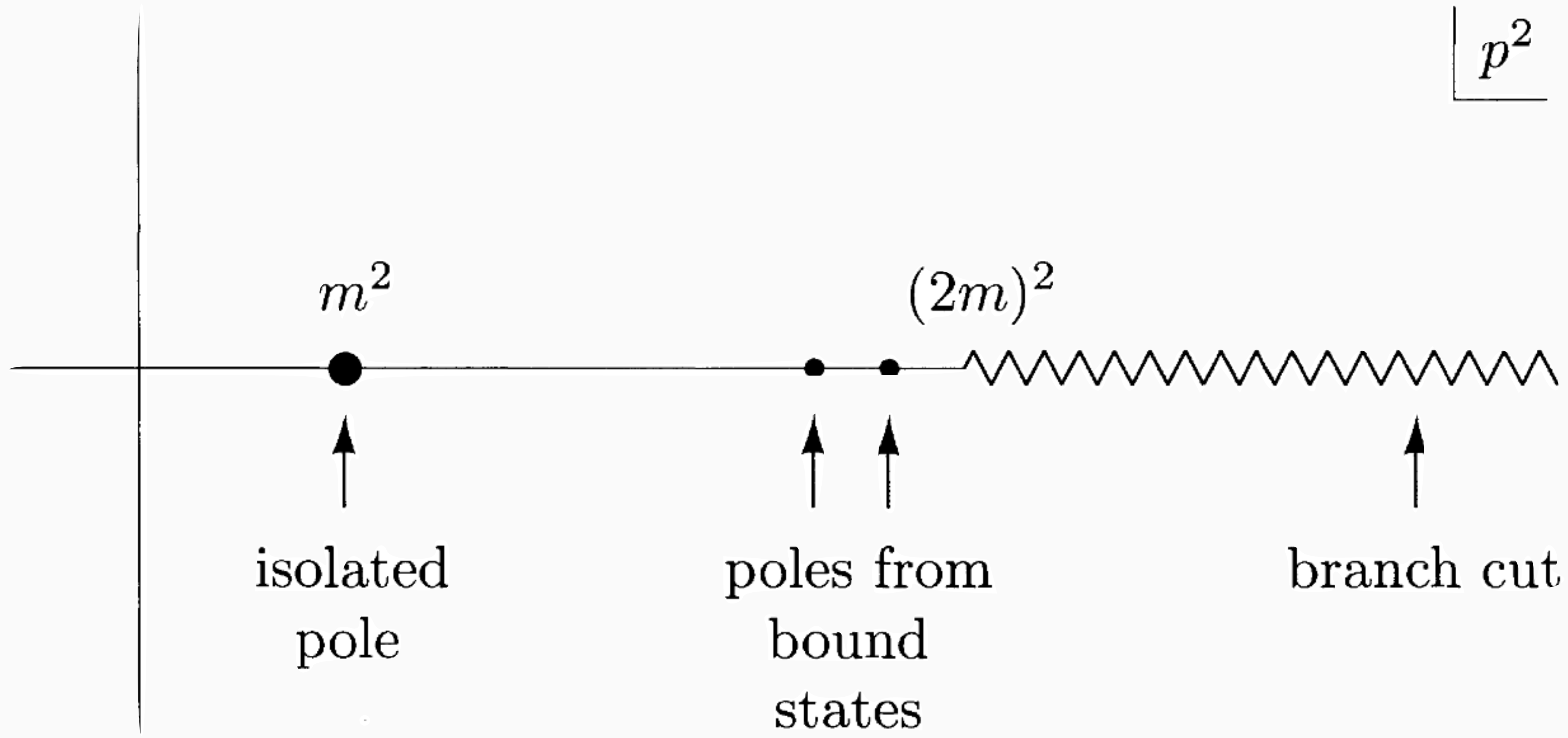


Figure 7.3. Analytic structure in the complex p^2-plane of the Fourier transform of the two-point function for a typical theory. The one-particle states contribute an isolated pole at the square of the particle mass. States of two or more free particles give a branch cut, while bound states give additional poles.

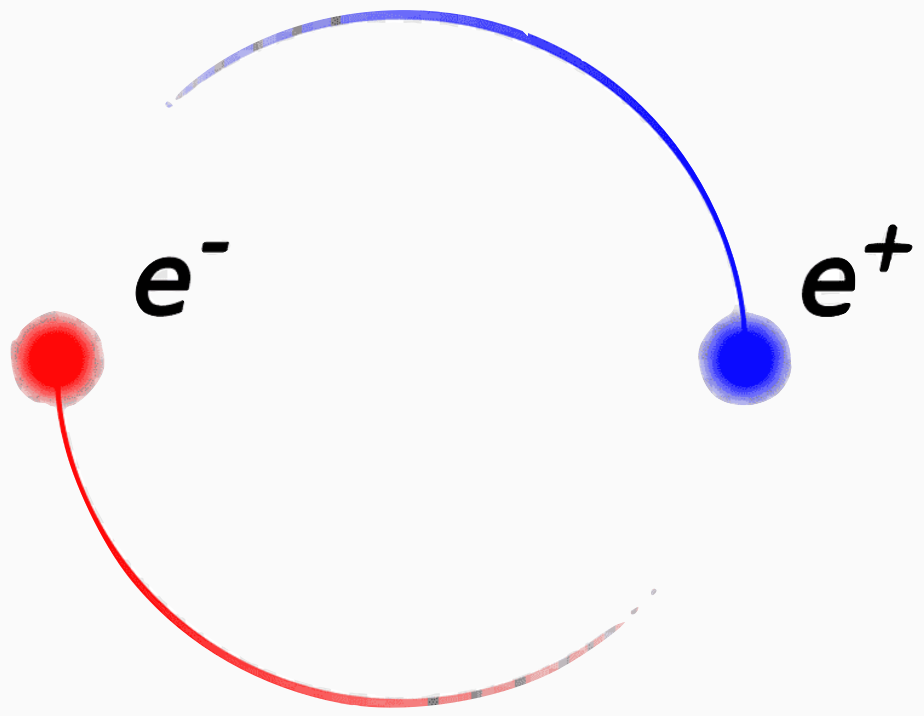
M. E. Peskin and D. V. Schroeder, An Introduction to quantum field theory, 1995

Can we have an intuitive description of bound states in relativistic case?

Positronium

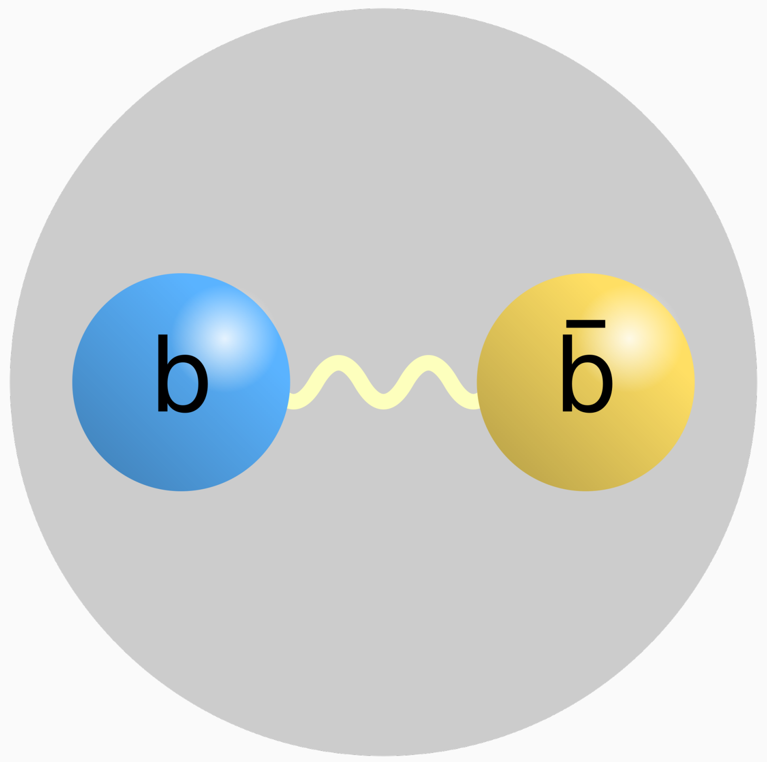
- A weak-coupled QED bound state: Positronium

Positronium



www.pngegg.com

Quarkonium



Cr. Avelludo

- In the center-of-mass (COM) frame, the kinetic energy is $\sim \alpha^2$, where $\alpha \approx \dfrac{1}{137}$ is the fine-structure constant, while the momentum is $\sim \alpha$, so no transverse photon at the leading order (LO);
- In the large-momentum frame, things become more complicated (dynamics);
- This work aims to develop an EFT method to study bound states in relativistic motion;
- It also provides hints on the heavy quarkonium system in QCD.

How to Describe a Bound State

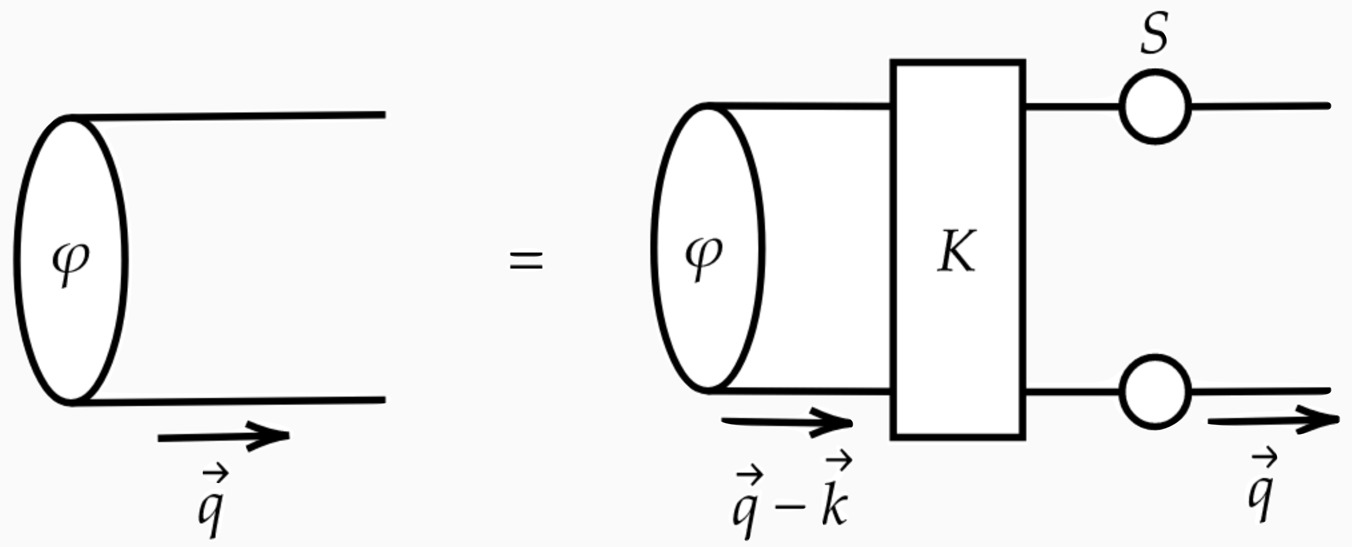
- **Non-relativistic case :**
  - **Wave function gives a complete description of a state in QM**
- **Relativistic case:**
  - **Particle number is frame-dependent**
    - **Hilbert space → Fock space**
  - **Lorentz symmetry: different methods have different degrees of violation of Lorentz symmetry**
    - **Bethe-Salpeter equation: defines covariant B-S amplitude as** $\Psi_{\mathbf{P}}(p)_{\alpha\beta} = \int d^4x e^{ix\cdot p} \langle \Omega | \mathrm{T} \left\{ \bar{\psi}_\beta(0) \psi_\alpha(x) \right\} | \mathbf{P}\lambda \rangle$

      E.E. Salpeter, H.A. Bethe, Phys.Rev. 84 (1951);
      W. Lucha, EPJ Web Conf. 274 (2022)
    - **Light-front quantization: encodes dynamics in boost-invariant variables**

      S. J. Brodsky, et al., Phys.Rept. 301 (1998)
    - **Fock state expansion with equal-time quantization: equal-time condition is frame-dependent**

How to Describe a Bound State in Relativistic Case

- We have some methods, but none of them is perfect
  - Bethe-Salpeter equation: covariant
    - Ladder approximation (single particle exchange)
    - Needs non-perturbative input in strong-coupling case, like propagator
  - Light-front quantization: boost invariant
    - Zero modes ($p^+ = 0$) are subtle and usually being omitted   X. Ji and Y. Liu, PRD 105 (2022)
    - Needs non-perturbative methods in strong-coupling case: Dyson-Schwinger equation, Lattice QCD, etc.
  - Fock state expansion with equal-time quantization: frame-dependent.
    - Solve the wave function in a specific frame
    - Not applicable to the theories with non-trivial vacuum structure or strong coupling, like QCD
    - But it is intuitive and compatible with the old-fashioned perturbation theory, so it is a good method to combine with EFT



S is single particle propagator, K is the kernel constructed with all possible two-particle irreducible diagrams.

Fock State Expansion

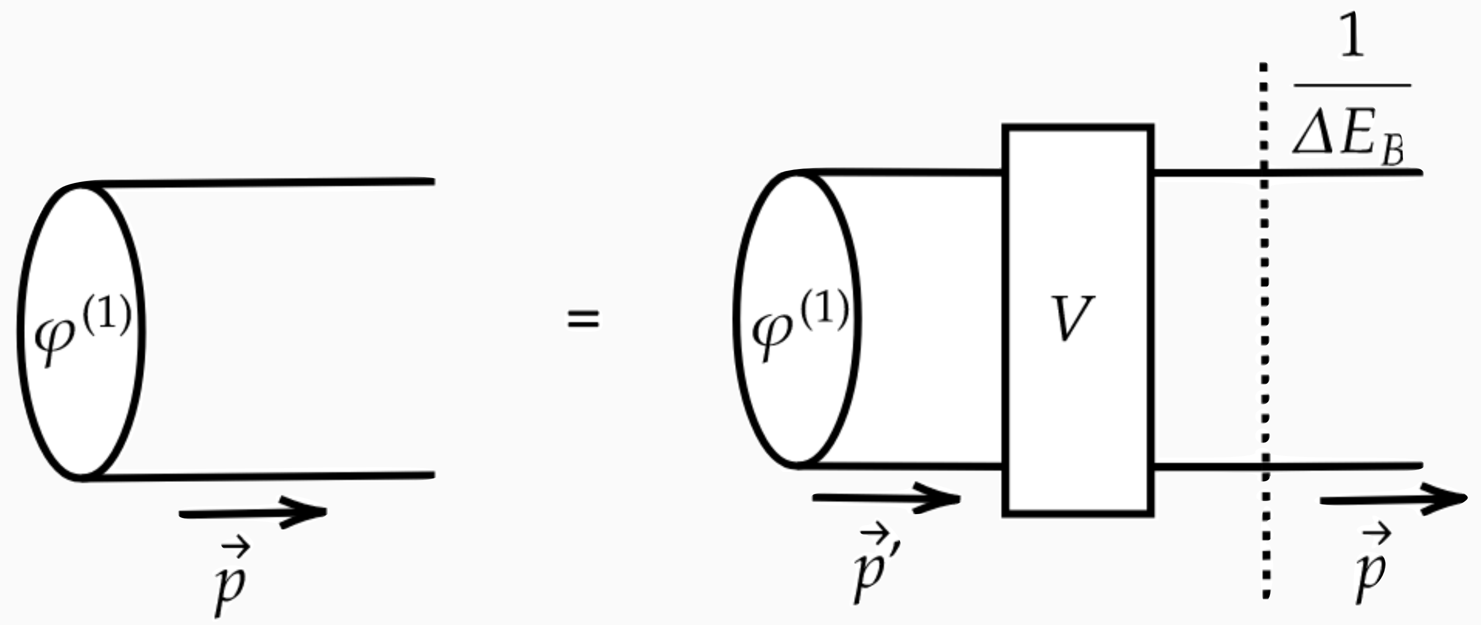
- **Fock state expansion of the positronium**
  - $$|\vec{P}\rangle_s \approx \sum_{s_1,s_2}\int \frac{d^3p}{(2\pi)^3} C^{(1)}_{s,s_1,s_2}(\vec{p}) \left| e^-_{s_1}(\vec{p}), e^+_{s_2}(\vec{P}-\vec{p}) \right\rangle + \sum_{s_1,s_2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3} C^{(2)}_{s,s_1,s_2}(\vec{p},\vec{k}) \left| e^-_{s_1}(\vec{p}), e^+_{s_2}(\vec{P}-\vec{p}-\vec{k}), \gamma(\vec{k}) \right\rangle + \ldots$$
  - **Wave function:** $C^{(1)}_{s,s_1,s_2}(\vec{p}) = \dfrac{\langle e^-_{s_1}(\vec{p}), e^+_{s_2}(\vec{P}-\vec{p}) \mid \vec{P}\rangle_s}{N} \equiv S^s_{s_1,s_2}(\vec{P})\varphi^{(1)}_{\vec{P}}(\vec{p})$

In the weak coupling limit, the pair creation and annihilation processes are suppressed, so spin part is not dynamical at LO.

- **Old-fashioned perturbation theory (OFPT)**
  - **Separate Hamiltonian as $H = H_0 + V$, suppose $|\psi\rangle$ are eigenstates of H, and $|\phi\rangle$ are eigenstates of H_0, then we have**

$$\langle \phi \mid \psi \rangle = \frac{1}{E_\psi - E_\phi}\sum_{\phi'} \frac{\langle \phi | V | \phi' \rangle}{\langle \phi \mid \phi \rangle}\langle \phi' \mid \psi \rangle \Longrightarrow \varphi^{(1)}(\vec{p}) = \frac{1}{E_\psi - E_\phi}\sum_{\phi'} \frac{\langle \phi | V | \phi' \rangle}{\langle \phi \mid \phi \rangle}\varphi^{(1)}(\vec{p}')$$

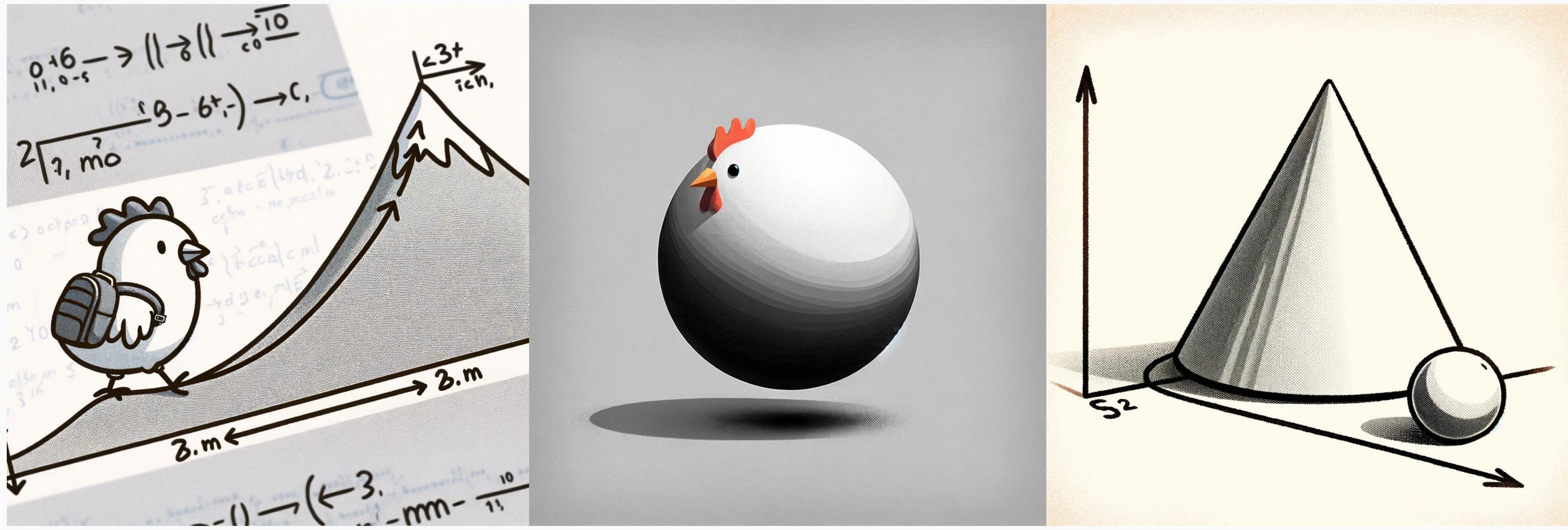
  - **The LO perturbation theory could be represented by a time-ordered diagram, where $\Delta E_B = E_\psi - E_\phi$**



EFT of Relativistic Positronium

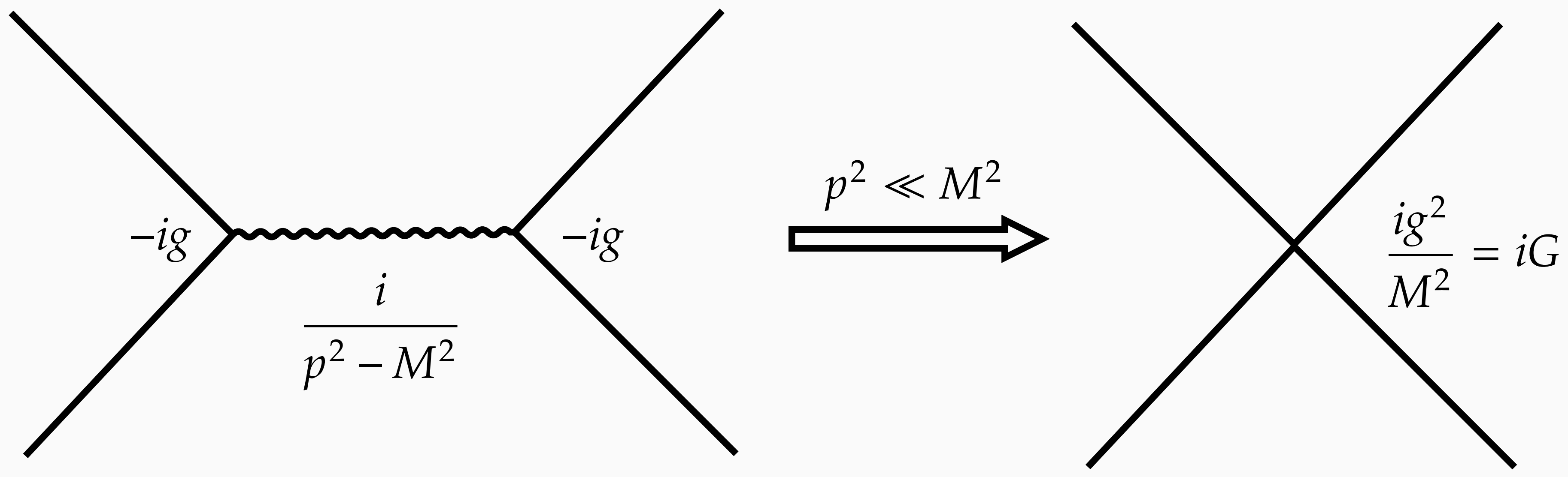
What is an Effective Field Theory

- When we analyze the motion of a chick climbing a hill using Newtonian mechanics…
- When we have a very heavy propagator…



Images created by ChatGPT + DALL-E

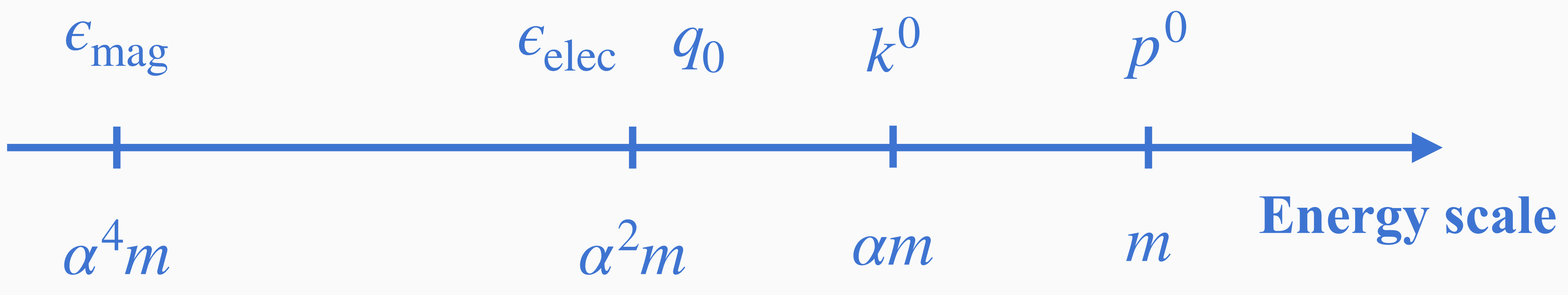
Four-fermion interaction



- Effective field theory (EFT)
  - To build up an EFT: degrees of freedom (only relevant ones), power counting (error estimation) and symmetries (no model dependence)
  - Two ways to construct EFTs: top-down (like NREFT), bottom-up (like Chiral P.T.)

Some EFTs can be constructed in both ways, like NRQED.

Simple Example: NRQED for Hydrogen atom in COM



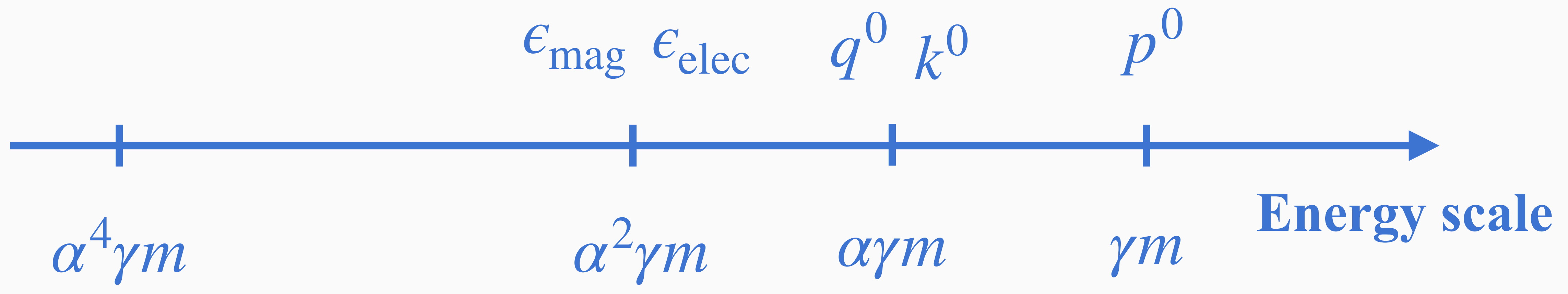
- **Degrees of freedom (under the Coulomb gauge)**
  - **Electron, Coulomb photon, radioactive photon**
- **Power counting**
  - **Fermion 4-momentum:** $p^\mu = mv^\mu + q^\mu \sim (1,0,0,0)m + (\alpha^2, \alpha, \alpha, \alpha)m$ **where** $p^2 = (mv+q)^2 = m^2\left(1 + \mathcal{O}(\alpha^2)\right)$
  - **Photon 4-momentum:** $k^\mu \sim (\alpha, \alpha, \alpha, \alpha)m$
  - **Energy:** $\epsilon = \sqrt{m^2 + |\vec{p}|^2} = m + \dfrac{|\vec{p}|^2}{2m} - \dfrac{|\vec{p}|^4}{8m^3} + \ldots \sim m + \alpha^2 m + \alpha^4 m + \ldots$
    - **Electric energy** $\epsilon_{\text{elec}} \sim \int d^3r |\vec{E}|^2 \sim \alpha^2 m$ **and magnetic energy** $\epsilon_{\text{mag}} \sim \int d^3r |\vec{B}|^2 \sim \alpha^4 m$
- **Symmetries**
  - **Gauge symmetry, 3D rotational symmetry** $SO(3)$**, Parity, Time reversal**

Simple Example: NRQED for Hydrogen atom in COM

- Bottom-up: The effective Lagrangian can be constructed by systematically listing all operators up to a given order that respect the symmetries of the theory.
- Gauge invariant / covariant ingredients: electric field $\overrightarrow{E}$, magnetic field $\overrightarrow{B}$, fermion spin $\vec{\sigma}$ and covariant derivative D^{μ}
- P-parity: $\overrightarrow{E}$ is odd, $\overrightarrow{B}$ is even, $\vec{\sigma}$ is even, $\overrightarrow{D}$ is odd;
- T-parity: $\overrightarrow{E}$ is even, $\overrightarrow{B}$ is odd, $\vec{\sigma}$ is odd, $\overrightarrow{D}$ is even;
- All possible Hermitian operators
  - Order m^0: iD^0 ;
  - Order m^{-1}: $|\overrightarrow{D}|^2/2m,\ \left(\vec{\sigma}\cdot\overrightarrow{B}\right)/2m$;
  - Order m^{-2}: $\left(\overrightarrow{D}\cdot\overrightarrow{E}-\overrightarrow{E}\cdot\overrightarrow{D}\right)/m^2,\ i\vec{\sigma}\cdot\left(\overrightarrow{D}\times\overrightarrow{E}-\overrightarrow{E}\times\overrightarrow{D}\right)/m^2$;
- The effective Lagrangian up to $\mathcal{O}(m^{-2})$

$$\mathscr{L}_{\mathrm{NRQED}}=\psi^{\dagger}\left\{iD^0+\frac{|\overrightarrow{D}|^2}{2m}+c_F\frac{e(\vec{\sigma}\cdot\overrightarrow{B})}{2m}+c_D\frac{e(\overrightarrow{D}\cdot\overrightarrow{E}-\overrightarrow{E}\cdot\overrightarrow{D})}{8m^2}+c_S\frac{ie\vec{\sigma}\cdot(\overrightarrow{D}\times\overrightarrow{E}-\overrightarrow{E}\times\overrightarrow{D})}{8m^2}\right\}\psi+\mathcal{O}(m^{-3})$$

EFT of Relativistic Positronium



- **Degrees of freedom (under the Coulomb gauge)**
  - **Electron, positron, Coulomb photon, radioactive photon**
- **Power counting (γ is the boost factor with hierarchy $\alpha\gamma^n \ll 1$, velocity $\beta \approx 1$ is dropped)**
  - **Fermion 4-momentum:** $p^\mu = mv^\mu + q^\mu \sim (\gamma,0,0,\gamma)m + (\alpha\gamma + \alpha^2\gamma, \alpha, \alpha, \alpha\gamma + \alpha^2\gamma)$ **and** $q_\perp^\mu \equiv q^\mu - (q\cdot v)v^\mu \sim (\alpha\gamma, \alpha, \alpha, \alpha\gamma)$
  - **Photon 4-momentum:** $k^\mu \sim (\alpha\gamma, \alpha, \alpha, \alpha\gamma)m$

    $q_\perp^2 \sim q^2 \sim \alpha^2 m^2$
    $(q\cdot v)^2 \sim \alpha^4 m^2$
  - **Electric energy** $\epsilon_{\mathrm{elec}} \sim \int d^3r\,|\vec{E}|^2 \sim \alpha^2\gamma m$ **and magnetic energy** $\epsilon_{\mathrm{mag}} \sim \int d^3r\,|\vec{B}|^2 \sim \alpha^2\gamma m$
- **Symmetries:**
  - **Gauge symmetry, 2D rotational symmetry $SO(2)$, Parity, Time reversal, Charge parity, Reprameterization symmetry**

EFT of Relativistic Positronium

- **Top-down: The effective Lagrangian can also be constructed from the full theory (QED)**

  - **QED Lagrangian:** $\mathscr{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

Separate spinor structure, recall the Dirac equation $(\not{p} - m)u^s(p) = 0$

Heavy component H_v **can be removed using EOM**

Separate momentum

  - **Effective field of electron:** $\psi_v(x) \equiv e^{imv\cdot x}\psi(x) = \left(\frac{1+\not{v}}{2} + \frac{1-\not{v}}{2}\right)\psi_v = h_v + H_v$

  - **Effective field of positron:** $\phi_v(x) \equiv e^{-imv\cdot x}\psi(x) = \left(\frac{1+\not{v}}{2} + \frac{1-\not{v}}{2}\right)\phi_v = X_v + \chi_v$

Or just treat it as another independent fermion field with opposite charge.

  - **Effective Lagrangian**

$D_\mu = \partial_\mu - ieA_\mu$

$\tilde{D}_\mu = \partial_\mu + ieA_\mu$

$D_\perp = D - (D\cdot v)v$

$$\mathscr{L}^{(1)}_{\text{eff}} = \bar{h}_v(iD\cdot v)h_v + \bar{h}_v\frac{(iD_\perp)^2}{2m}h_v + \bar{\chi}_v(i\tilde{D}\cdot v)\chi_v + \bar{\chi}_v\frac{(i\tilde{D}_\perp)^2}{2m}\chi_v - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Electron and positron are decoupled.

EFT of Relativistic Positronium

- **Effective Hamiltonian (electron only)**

$$\mathscr{H}_h^{(1)} = \bar{h}\left(i\vec{D}\cdot\vec{v} + eA^0v^0\right)h + \bar{h}\frac{\left(i\partial^0\right)^2 - \left(eA^0\right)^2}{2m}h + \bar{h}\frac{\left(i\vec{D}\right)^2}{2m}h + \bar{h}\frac{\left(i\vec{D}\cdot\vec{v} + eA^0v^0\right)^2 - \left(i\partial^0v^0\right)^2}{2m}h$$

- **Recall the power counting**

e	A^0	$\lvert\vec{A}\rvert$	v^0	$\lvert\vec{v}\rvert$	$\lvert\vec{q}\rvert$
$\alpha^{1/2}$	$\alpha^{3/2}\gamma$	$\alpha^{3/2}\gamma$	γ	γ	$\alpha\gamma$

- **The interaction terms (electron only) in the effective Hamiltonian is**

$$\begin{aligned}\mathscr{H}_{\text{vertex}}^{(1)} = \ & \bar{h}\left(eA^0v^0 - e\vec{A}\cdot\vec{v}\right)h & \text{-------} \quad \alpha^2 \\ & -\bar{h}\frac{e\vec{A}\cdot(\vec{q}+\vec{q}\,')}{2m}h - \bar{h}\frac{(e\vec{A}\cdot\vec{v})(\vec{q}+\vec{q}\,')\cdot\vec{v}}{2m}h & \text{-------} \quad \alpha^3 \\ & -\bar{h}\frac{\left(eA^0\right)^2}{2m}h + \bar{h}\frac{(e\vec{A})^2}{2m}h + \bar{h}\frac{(e\vec{A}\cdot\vec{v})^2}{2m}h + \bar{h}\frac{\left(eA^0v^0\right)^2}{2m}h - \bar{h}\frac{e^2A^0v^0\vec{A}\cdot\vec{v}}{2m}h\,. & \text{------} \quad \alpha^4\end{aligned}$$

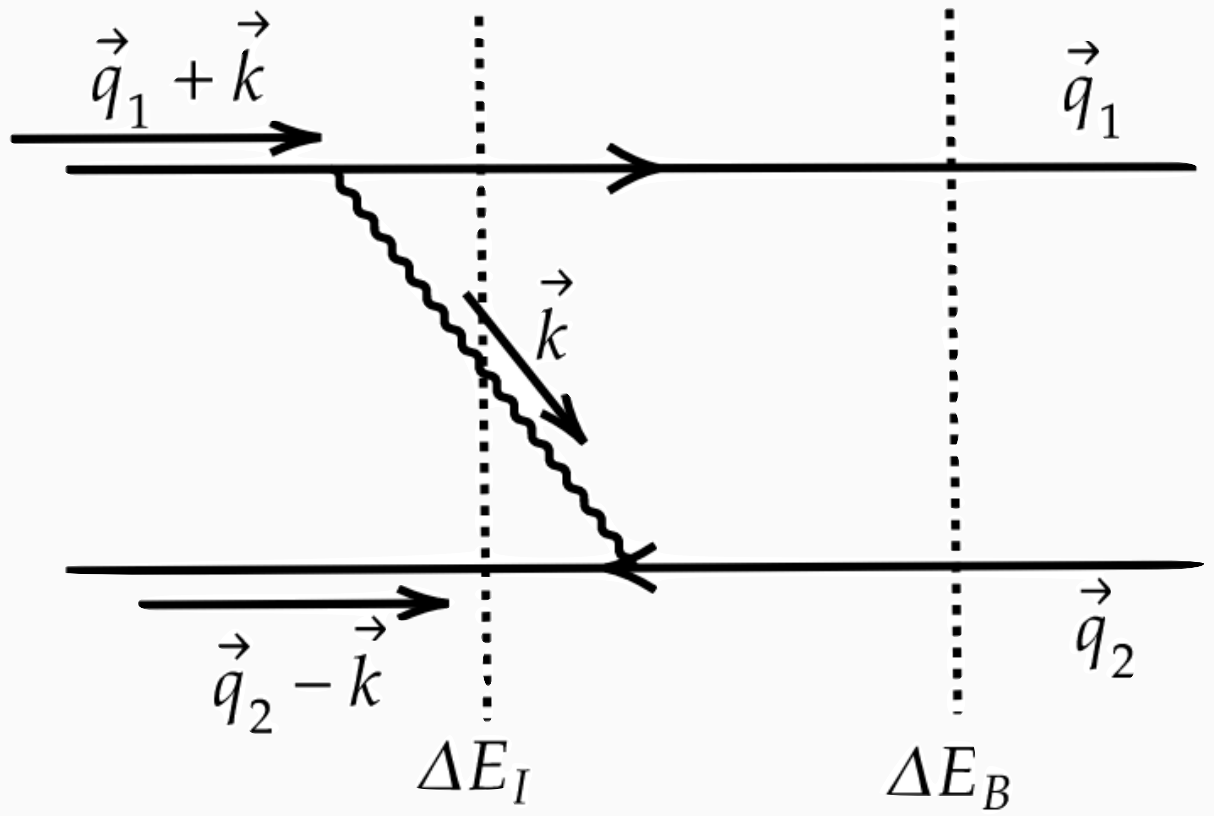
Energy Denominator

- **Old-fashioned perturbation theory (OFPT) at LO**

$$\varphi^{(1)}(\vec{p}) = \frac{1}{E_\psi - E_\phi} \sum_{\phi'} \frac{\langle \phi | V | \phi' \rangle}{\langle \phi \mid \phi \rangle} \varphi^{(1)}(\vec{p}')$$

  - **The power counting of time-ordered diagrams is related to both the interaction terms and the energy denominator.**

- **There are two kinds of energy denominators potentially contribute at leading order**



  - **Intermediate states without photon:** $\Delta E_B = E_{\vec{P}} - E_{\vec{p}_1} - E_{\vec{p}_2} \sim \alpha^2 \gamma^{-1} m$

When $\gamma \to \infty$, the binding energy $\Delta E_B \to 0$, just like the parton model.

  - **Intermediate states with one or more photons:** $\Delta E_I = E_{\vec{P}} - E_{\vec{p}_1} - E_{\vec{p}_2 - \vec{k}} - |\vec{k}| \sim \alpha \gamma^{-1} m$

- **Ultra-soft photon:** $|\vec{k}| \sim \alpha^2 \gamma m$ **and** $\Delta E_I \sim \alpha^2 \gamma^{-1} m$ **, we just need to count the power of** $|\vec{k}|$ **and** ΔE_I **, named** N_k

Solving Positronium in COM Frame

Apply EFT in Static Positronium

- **The effective Hamiltonian in the static case ($v^\mu = (1,0,0,0)$) is**

$$\mathscr{H}_{\text{eff}}^{(1)} = h_v^\dagger \left(eA^0 + \frac{(i\vec{D})^2}{2m} \right) h_v + \chi_v^\dagger \left(-eA^0 + \frac{(i\tilde{\vec{D}})^2}{2m} \right) \chi_v + \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$$

- **The power counting in the static case**

e	A^0	$\lvert\vec{A}\rvert$	v^0	$\lvert\vec{v}\rvert$	$\lvert\vec{q}\rvert$
$\alpha^{1/2}$	$\alpha^{3/2}$	$\alpha^{5/2}$	1	0	α

- **Using the old-fashioned perturbation theory**

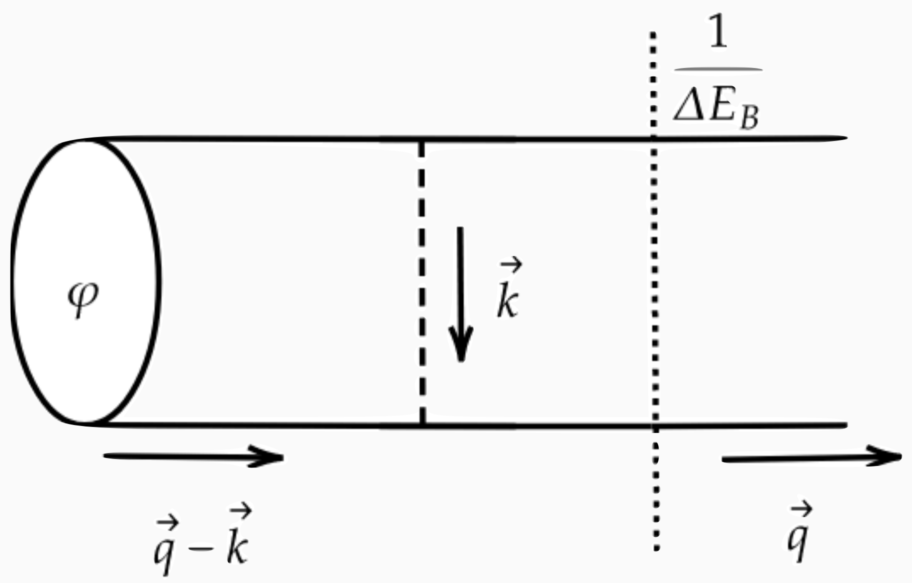
$$\langle \phi \mid \psi \rangle = \frac{1}{E_\psi - E_\phi} \sum_{\phi'} \frac{\langle \phi \,|\, V \,|\, \phi' \rangle}{\langle \phi \mid \phi \rangle} \langle \phi' \mid \psi \rangle$$

  - **Order α^2 :** $V_1 = h_v^\dagger \left(eA^0\right) h_v + \chi_v^\dagger \left(-eA^0\right) \chi_v$
  - **Order α^3 :** $V_2 = h_v^\dagger \left(\frac{-ie\vec{q} \cdot \vec{A}}{2m} \right) h_v + \chi_v^\dagger \left(\frac{ie(-\vec{q}) \cdot \vec{A}}{2m} \right) \chi_v$

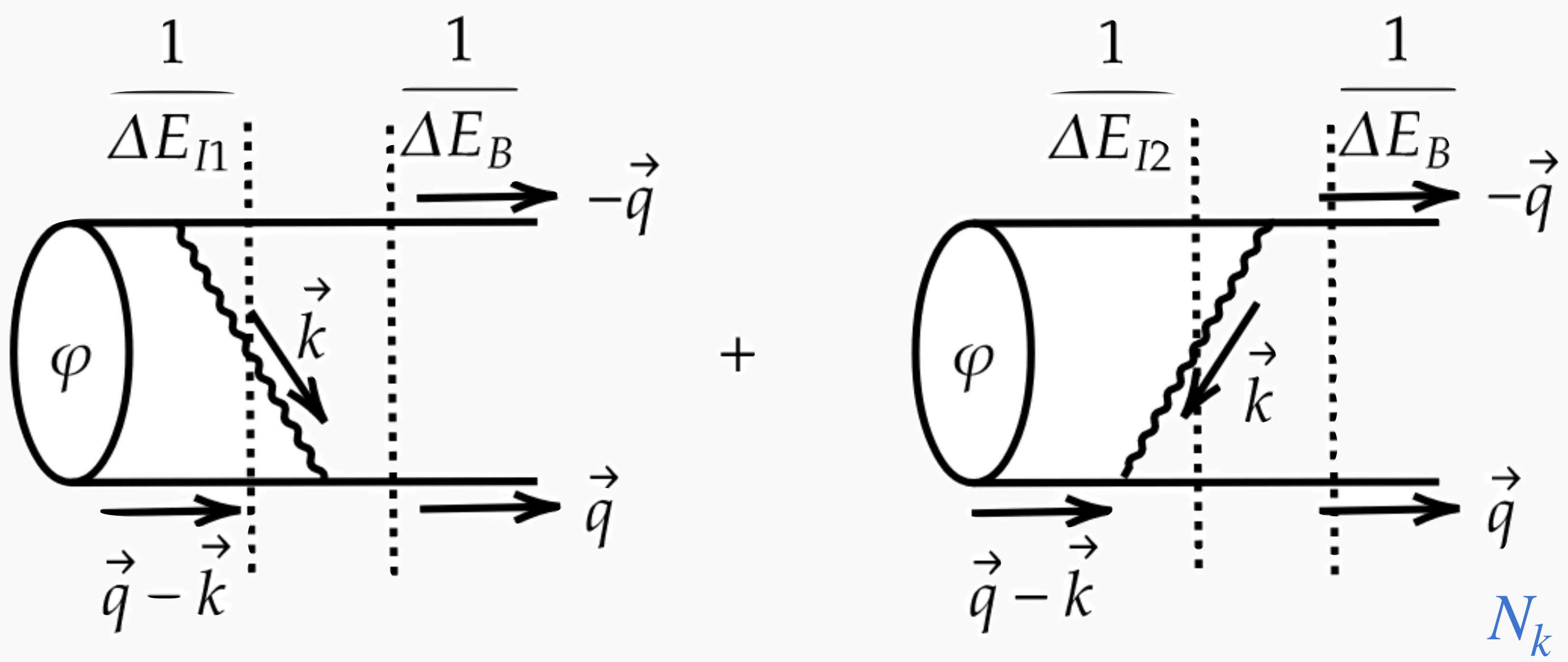
Solving Static Positronium with OFPT

- Consider the first two orders in the effective potential
  - Order α^2 : $V_1 = h_v^\dagger \left(eA^0\right) h_v + \chi_v^\dagger \left(-eA^0\right) \chi_v = h_v^\dagger \chi_v^\dagger \left(\frac{-e^2}{|\vec{k}|^2}\right) h_v \chi_v$ because $\nabla^2 A^0 = -\rho = -e\psi^\dagger \psi$

    Here we ignored the self-energy.
  - Order α^3 : $V_2 = h_v^\dagger \left(\frac{-ie\vec{q}\cdot\vec{A}}{2m}\right) h_v + \chi_v^\dagger \left(\frac{ie(-\vec{q})\cdot\vec{A}}{2m}\right) \chi_v$
- There are three time-ordered diagrams



$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{\Delta E_B} \frac{-e^2}{|\vec{k}|^2} \sim \alpha^{-1} |\vec{k}|$$



$N_k = 1$, so the ultra-soft photon will not break the power counting.

$$\int \frac{d^3k}{(2\pi)^3} \frac{D_{ij}}{\Delta E_B} e^2 \frac{-q_i q_j}{(2m)^2} \left(\frac{1}{\Delta E_{I1}} + \frac{1}{\Delta E_{I2}}\right) \sim \alpha |\vec{k}|$$

$$D_{ij} = \left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}\right) \frac{1}{2|\vec{k}|}$$

Wave Function of Static Positronium

- **Using the OFPT**

$$\langle \phi \mid \psi \rangle = \frac{1}{E_\psi - E_\phi} \sum_{\phi'} \frac{\langle \phi \,|\, V \,\big|\, \phi' \rangle}{\langle \phi \mid \phi \rangle} \langle \phi' \mid \psi \rangle$$

- **The wave function satisfies**

$$\varphi_{\vec{0}}\left(\vec{q}_2\right) = \frac{1}{\Delta E_B} \int \frac{d^3k}{(2\pi)^3} \left(\frac{-e^2}{|\vec{k}|^2} \right) \varphi_{\vec{0}}\left(\vec{q}_2 - \vec{k}\right) = \frac{-e^2}{\Delta E_B} \int \frac{d^3k}{(2\pi)^3} \frac{1}{|\vec{k}|^2} \varphi_{\vec{0}}\left(\vec{q}_2 - \vec{k}\right)$$

  - **Using convolution theorem, it gives the Coulomb's Law**

$$\Delta E_B \, \tilde{\varphi}_{\vec{0}}(\vec{x}) = \left(\frac{-e^2}{4\pi |\vec{x}|} + \mathcal{O}(\alpha^4) \right) \tilde{\varphi}_{\vec{0}}(\vec{x})$$

Solving Positronium in Relativistic Motion

OFPT of Relativistic Positronium

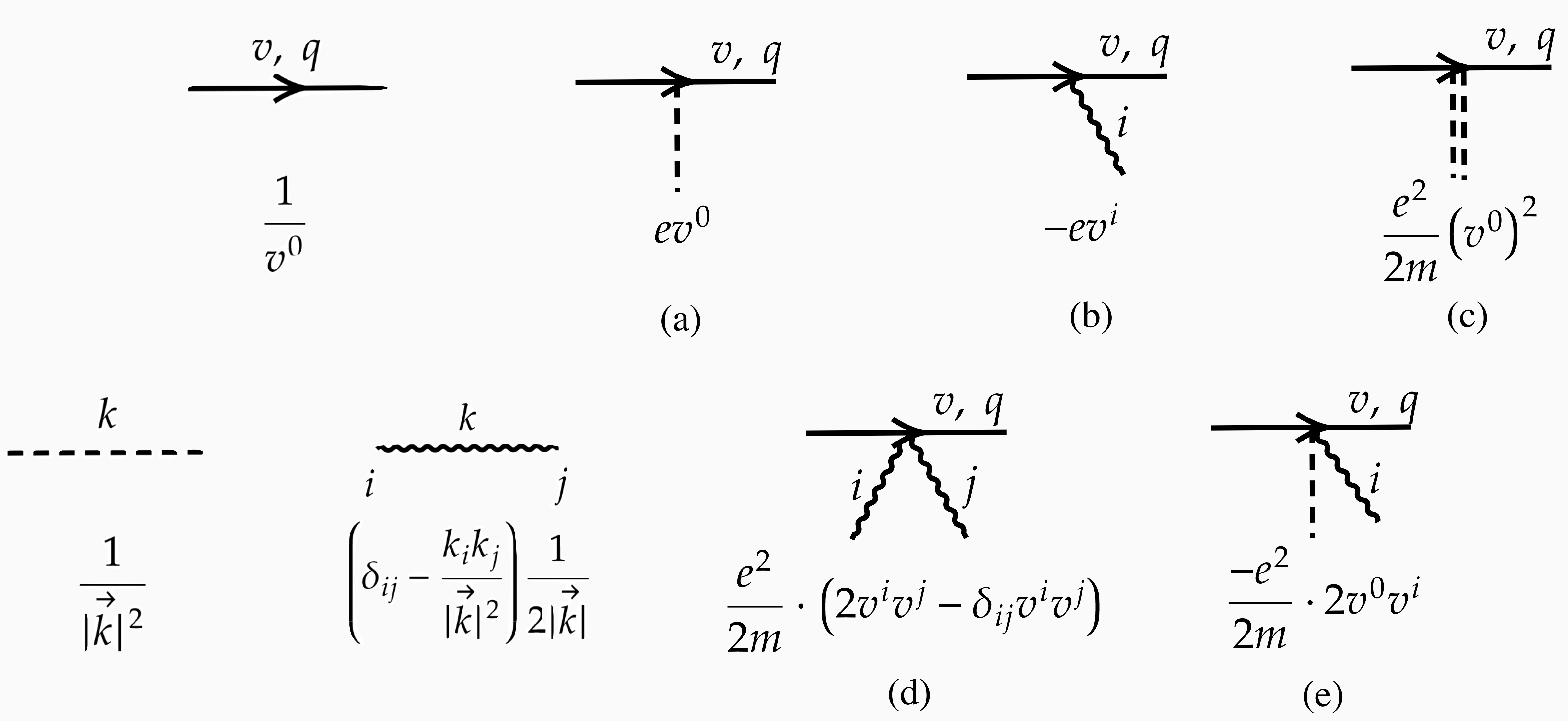
- **The interaction terms (electron only) in the effective Hamiltonian is**

$$\mathscr{H}^{(1)}_{\text{vertex}} = \bar h\underbrace{\left(eA^0v^0 - e\vec{A}\cdot\vec{v}\right)}_{\alpha^2}h - \bar h\underbrace{\frac{e\vec{A}\cdot(\vec{q}+\vec{q}\,')}{2m}h - \bar h\frac{(e\vec{A}\cdot\vec{v})(\vec{q}+\vec{q}\,')\cdot\vec{v}}{2m}}_{\alpha^3}h$$

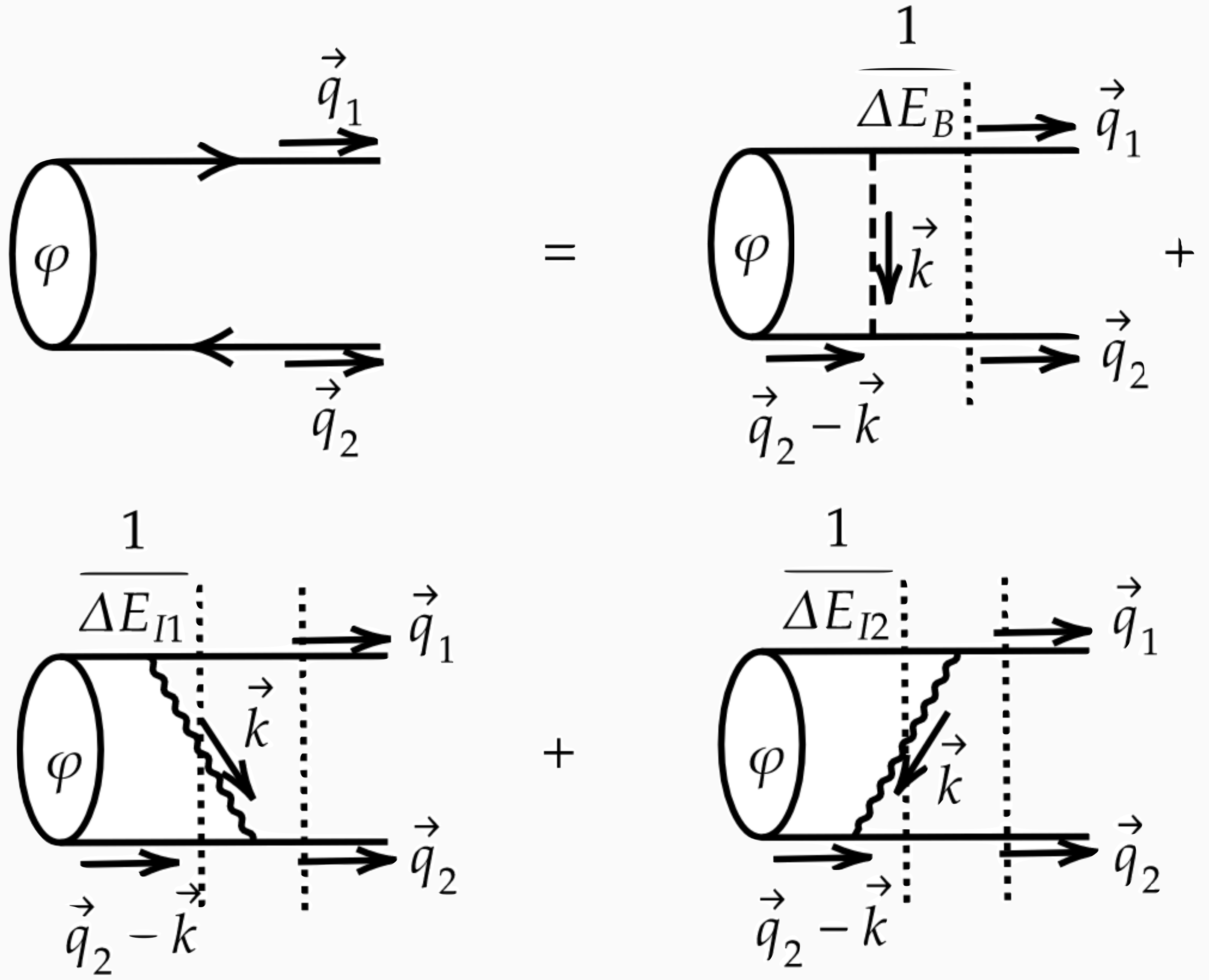
$$\underbrace{-\bar h\frac{(eA^0)^2}{2m}h + \bar h\frac{(e\vec{A})^2}{2m}h + \bar h\frac{(e\vec{A}\cdot\vec{v})^2}{2m}h + \bar h\frac{(eA^0v^0)^2}{2m}h - \bar h\frac{e^2A^0v^0\vec{A}\cdot\vec{v}}{2m}h}_{\alpha^4}\,.$$

- **The LO terms of vertices are $eA^0v^0 - e\vec{A}\cdot\vec{v}$, using OFPT we have three time-ordered diagrams**

Feynman rules of vertices



Equation of wave function at LO



Wave Function of Relativistic Positronium

- **The wave function of the relativistic positronium satisfies**

$$\varphi_{\vec{Q}}\left(\vec{q}_2\right) = \frac{1}{\Delta E_B}\left(\frac{1}{v^0}\right)^2 \int \frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{-e^2}{|\vec{k}|^2}\left(v^0\right)^2 + \frac{1}{\delta E}\left[|\vec{v}|^2 - \frac{(\vec{v}\cdot\vec{k})^2}{|\vec{k}|^2}\right]\frac{-e^2}{2|\vec{k}|}\right)\varphi_{\vec{Q}}\left(\vec{q}_2 - \vec{k}\right)$$

$$= \frac{-e^2}{\Delta E_B}\int \frac{d^3\vec{k}}{(2\pi)^3}\left[\frac{1}{|\vec{k}|^2} + \frac{\beta^2}{\delta E}\cdot\frac{|\vec{k}_\perp|^2}{2|\vec{k}|^3}\right]\varphi_{\vec{Q}}\left(\vec{q_2} - \vec{k}\right)$$

- **The energy denominator is**

$$\frac{1}{\delta E} = \frac{1}{E - E_{\vec{q}_1} - E_{\vec{q}_2-\vec{k}} - |\vec{k}|} + \frac{1}{E - E_{\vec{q}_2} - E_{\vec{q}_1+\vec{k}} - |\vec{k}|} = \frac{1}{\Delta E_I} + \frac{1}{\Delta E_I + E_{\vec{q}_1} + E_{\vec{q}_2-\vec{k}} - E_{\vec{q}_2} - E_{\vec{q}_1+\vec{k}}} = \frac{-2|\vec{k}|}{|\vec{k}|^2 - (\vec{\beta}\cdot\vec{k})^2}$$

- **Then the equation can be simplified as**

$$\varphi_{\vec{Q}}(\vec{q}_2) = \frac{-e^2}{\Delta E_B}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{|\vec{k}|^2}\left[1 - \frac{\beta^2|\vec{k}_\perp|^2}{|\vec{k}|^2 - (\vec{\beta}\cdot\vec{k})^2}\right]\varphi_{\vec{Q}}(\vec{q}_2 - \vec{k}) = \frac{-e^2}{\Delta E_B}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\gamma^2|\vec{k}_\perp|^2 + k_\parallel^2}\varphi_{\vec{Q}}(\vec{q}_2 - \vec{k})$$

Wave Function of Relativistic Positronium

- **Change momentum variables**

$$\varphi_{\vec{Q}}(\vec{q}_2) = \frac{-e^2}{\Delta E_B} \int \frac{\gamma d^3\hat{k}}{(2\pi)^3} \frac{1}{\gamma^2(\hat{k}_\perp^2 + \hat{k}_\parallel^2)} \varphi_{\vec{Q}}(\vec{q}_2 - \vec{k}), \text{ where } \hat{k}_\parallel = k_\parallel/\gamma,\ \hat{k}_\perp^2 = |\vec{k}_\perp|^2,\ \hat{q}_\parallel = q_\parallel/\gamma,\ \hat{q}_\perp^2 = |\vec{q}_\perp|^2$$

- **The wave function of the relativistic positronium satisfies**

$$\varphi_{\vec{Q}}(\vec{q}_2) = \frac{-e^2}{\gamma \Delta E_B} \int \frac{d^3\hat{k}}{(2\pi)^3} \frac{1}{(\hat{k}_\perp^2 + \hat{k}_\parallel^2)} \varphi_{\vec{Q}}(\vec{q}_2 - \vec{k})$$

Note that $\Delta E_B \sim \alpha^2 \gamma^{-1} m$, so $\gamma \Delta E_B = \Delta E_{B0}$ in COM.

  - **It has the same form as the equation in the static case, so we have**

$$\varphi_{\vec{Q}}(\vec{q}) = \frac{1}{\sqrt{\gamma}} \varphi_{\vec{0}}(\hat{q}), \text{ with normalization } \int \frac{d^3\vec{q}}{(2\pi)^3} \varphi_{\vec{Q}}^*(\vec{q}) \varphi_{\vec{Q}}(\vec{q}) = \int \frac{d^3\hat{q}}{(2\pi)^3} \varphi_{\vec{0}}^*(\hat{q}) \varphi_{\vec{0}}(\hat{q})$$

- **It is found that the LO wave function of the relativistic positronium exactly contracts in the direction of motion.**

- **However, this is a non-trivial result because of the dynamical effects, the contribution of transverse photon becomes LO.**

- **To show the dynamical effects explicitly, we will evaluate the photon momentum distribution in the following slides.**

Photon Momentum Operator

- To construct a gauge invariant photon momentum operator, we can make use of the energy-momentum tensor

$$\hat{P}^i(t) = \int d^3x T^{0i}(\vec{x},t) = \int d^3x \left(-F^{0\alpha}(\vec{x},t)F^i_\alpha(\vec{x},t)\right) = -\int \frac{d^3k}{(2\pi)^3} F^{0\alpha}(-\vec{k},t)F^i_\alpha(\vec{k},t)$$

  - Then we can define the photon momentum operator as

$$\hat{p}_{i,\mathrm{ph}}(\vec{k}) \equiv -\frac{1}{2(2\pi)^3}\left(F^{0\alpha}(-\vec{k})F^i_\alpha(\vec{k}) + \mathrm{h.c.}\right)$$

  - The photon momentum distribution in a moving positronium $|\vec{Q}\rangle_s$ (effective momentum $\vec{Q} \equiv \vec{P} - 2m\vec{v}$) is

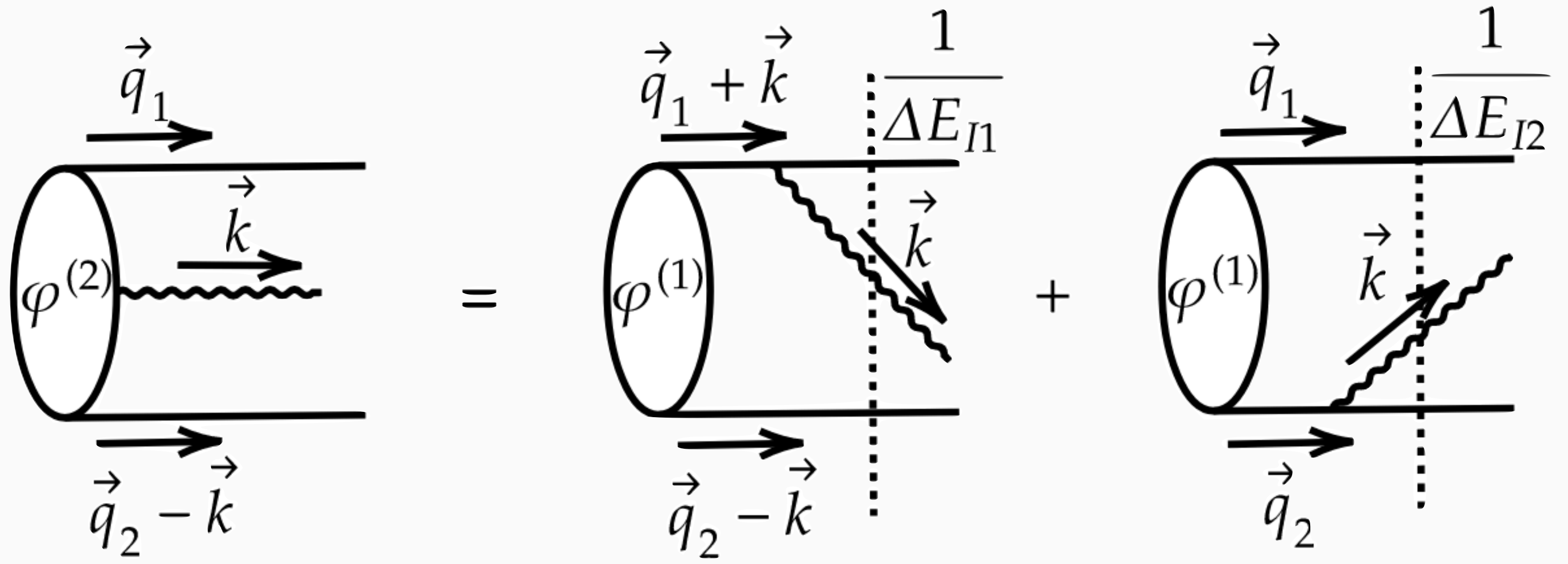
$$G_{Q,i}(\vec{k}) \equiv \left\langle \hat{p}_i(\vec{k}) \right\rangle = -\frac{1}{2(2\pi)^3}\int \frac{dk^0 dk'^0}{(2\pi)^2} {}_s\langle \vec{Q}| \left(F^{0\alpha}\left(-k^0, -\vec{k}\right)F^i_\alpha\left(k'^0, \vec{k}\right) + \mathrm{h.c.}\right)|\vec{Q}\rangle_s$$

  - Using the wave function that we solved before

$$\langle \vec{Q}|\hat{p}_i(\vec{k})|\vec{Q}\rangle = \int \frac{d^3q'}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3} \varphi^{(2)*}_{\vec{Q}}\left(\vec{q}',\vec{k}'\right)\varphi^{(2)}_{\vec{Q}}(\vec{q},\vec{k})\left\langle e^-_{s_1}, e^+_{s_2}, \gamma\left(\vec{k}'\right) | \hat{p}_i(\vec{k}) | e^-_{s_1}, e^+_{s_2}, \gamma(\vec{k})\right\rangle$$

  - The NLO wave function can be expanded in terms of the LO wave function

$$\varphi^{(2)}_{\vec{Q}}\left(\vec{q}_2,\vec{k}\right) = \frac{1}{\Delta E_I}\int \frac{d^3q'}{(2\pi)^3}\frac{\langle e^-_{s_1}(\vec{q}_1), e^+_{s_2}(\vec{q}_2-\vec{k}), \gamma(\vec{k})|V|e^-_{s_1}(\vec{Q}-\vec{q}'), e^+_{s_2}(\vec{q}')\rangle}{N}\varphi^{(1)}_{\vec{Q}}\left(\vec{q}'\right)$$

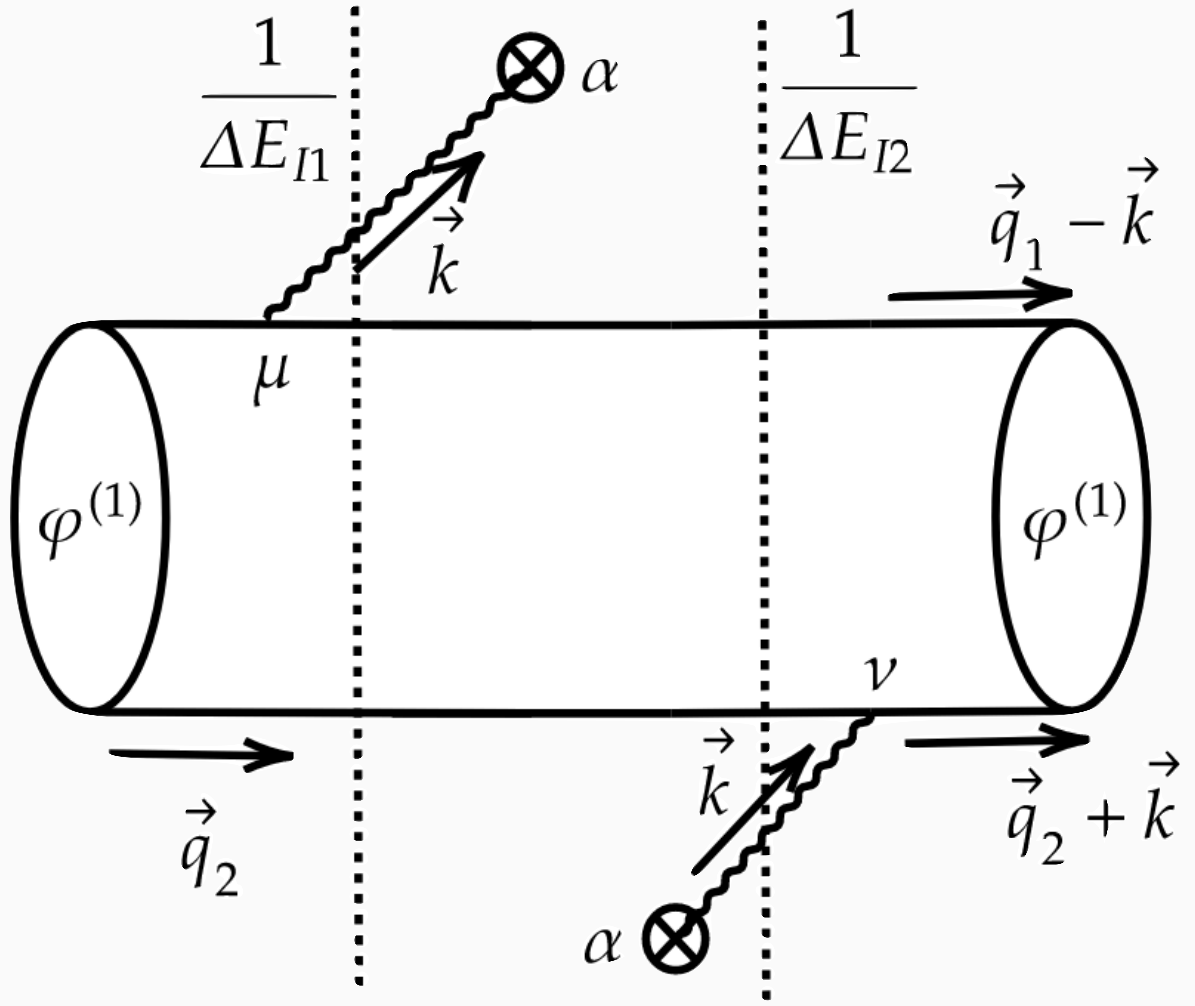


Photon Momentum Distribution

- **We know that**

$$F^{0\alpha}(-\vec{k})F^{i}_{\alpha}(\vec{k}) = k^0k^i\tilde{A}^{\alpha}(-\vec{k})\tilde{A}_{\alpha}(\vec{k})+k^{\alpha}k_{\alpha}\tilde{A}^{0}(-\vec{k})\tilde{A}^{i}(\vec{k})$$

- **The first term above can be expressed in terms of four connected time-ordered diagrams, one of which is**



  - **Using the Feynman rules, we have the expression**

$$G_1\left(\vec{k},\vec{q}_2\right) = \frac{-1}{2(2\pi)^3}\frac{1}{\Delta E_{I1}}\frac{1}{\Delta E_{I2}}\frac{1}{\left(\nu^0\right)^2}\varphi^{*}_{\vec{Q}}\left(\vec{q}_2\right)\left[k^0k^iD^{\alpha\mu}D^{\nu}_{\alpha}(-e)v_{\mu}(e)v_{\nu}\right]\varphi_{\vec{Q}}\left(\vec{q}_2+\vec{k}\right)$$

  - **Similarly, the second term has contribution**

$$G_2\left(\vec{k},\vec{q}_2\right) = \frac{-1}{2(2\pi)^3}\frac{1}{\Delta E_{I}}\frac{1}{\left(\nu^0\right)^2}\varphi^{*}_{\vec{Q}}\left(\vec{q}_2\right)\left[|\vec{k}|^2D^{00}D^{i\nu}(e)v_0(e)v_{\nu}\right]\varphi_{\vec{Q}}\left(\vec{q}_2+\vec{k}\right)$$

Photon Momentum Distribution

- The photon momentum distribution in a relativistic positronium is

Note the factor of 4 comes from h.c. term and symmetry factor of the diagrams.

$$G_{Q,\parallel}(\vec{k}) = \frac{\alpha}{\pi^2}\frac{1}{(\beta^2 k_\parallel^2 - |\vec{k}|^2)^2}\beta|\vec{k}_\perp|^2 \int \frac{d^3\vec{q}_2}{(2\pi)^3} |\varphi_{\vec{Q}}(\vec{q}_2) - \varphi_{\vec{Q}}(\vec{q}_2 + \vec{k})|^2$$

- If we take the light-cone limit $\gamma \to \infty$ and $\beta \to 1$, it is consistent with the result in literature

M. Burkardt, Nucl.Phys.B 373 (1992)

$$G_{Q,\parallel}(x, \hat{k}_\perp) = \frac{\alpha}{\pi^2}\frac{\hat{k}_\perp^2}{\left((\hat{k}_\perp^2)^2 + (2mvx)^2\right)^2} \int \frac{d^2\hat{q}_{2\perp}}{(2\pi)^2} \int \frac{dy}{2\pi} |\varphi_{\vec{0}}(y, \hat{q}_{2\perp}) - \varphi_{\vec{0}}(y, \hat{q}_{2\perp} + \hat{k}_\perp)|^2 \ ,$$

- We know the COM wave function of positronium

$$\varphi_{\vec{Q}}(\vec{q}) = \frac{1}{\sqrt{\gamma}}\varphi_{\vec{0}}(\hat{q}) = \frac{1}{\sqrt{\gamma}}\sqrt{\frac{512\pi}{\alpha^3 m^3}}\left[1 + \left(\frac{2|\hat{q}|}{\alpha m}\right)^2\right]^{-2}$$

- The final results in the momentum space is

$$G_{Q,\parallel}(\hat{k}) = \frac{2\alpha}{\pi^2}\frac{1}{(\hat{k}_\parallel^2 + \hat{k}_\perp^2)^2}\beta\hat{k}_\perp^2\left[1 - \left(1 + \frac{|\hat{k}|^2}{\alpha^2 m^2}\right)^{-2}\right]$$

It is proportional to β, so it vanishes in $\beta \to 0$.

Photon Momentum Distribution

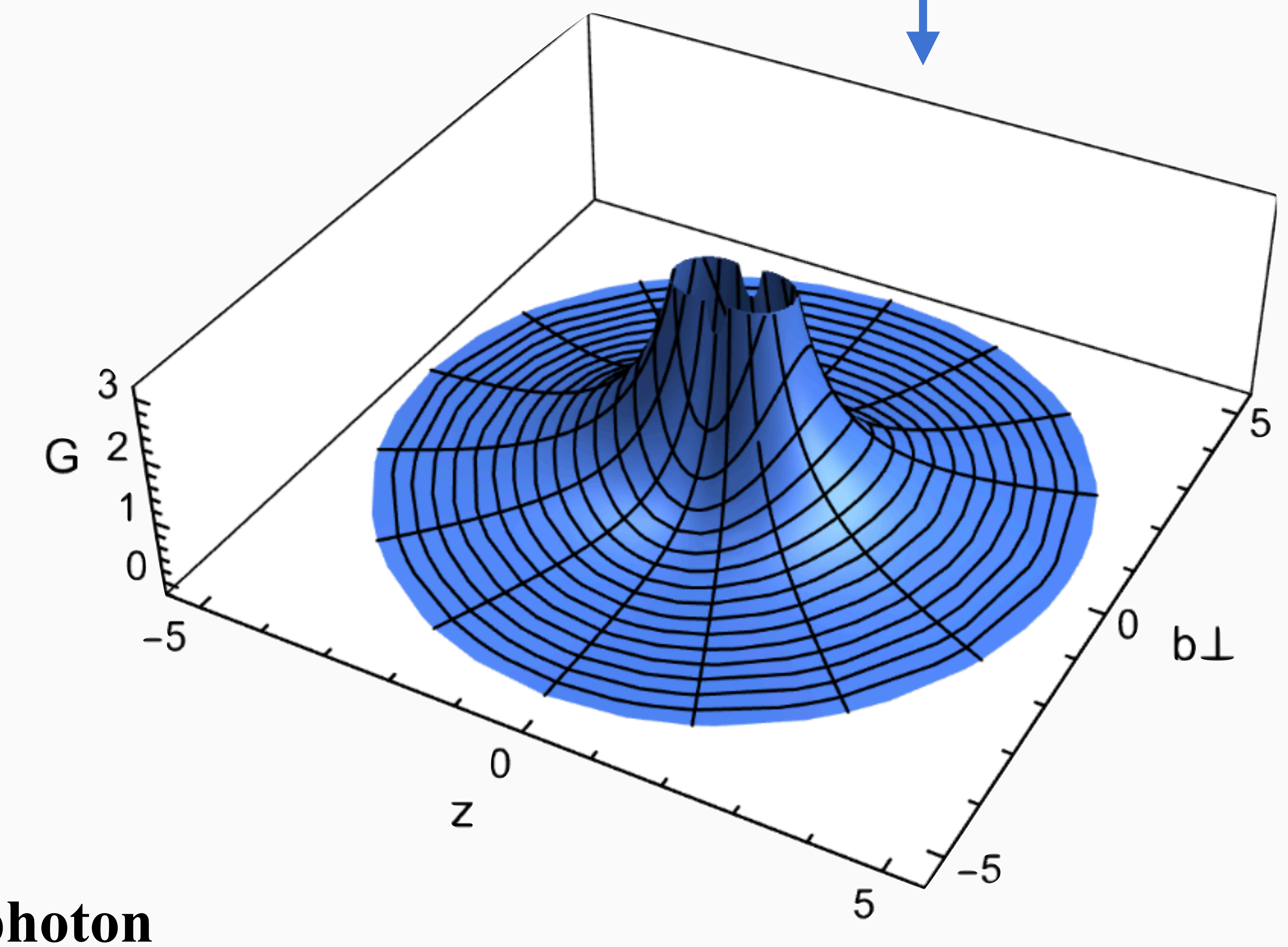
- Fourier transform to the coordinate space and rescale the conjugate coordinate of $\hat{k}$ as $\hat{r} \equiv \alpha m \vec{r} = (\hat{b}_\perp, \hat{z})$

$$\frac{(2\pi)^3}{\beta\alpha^2 m}\tilde{G}_{Q,\parallel}(\hat{r}) = \frac{16\hat{z}^2 - 8|\hat{b}_\perp|^2}{|\hat{r}|^5} + \frac{2e^{-|\hat{r}|}}{|\hat{r}|^5}\left[|\hat{b}_\perp|^4(|\hat{r}|+3) + |\hat{b}_\perp|^2(\hat{z}^2+4)(|\hat{r}|+1) - 2\hat{z}^2(4|\hat{r}| + \hat{z}^2 + 4)\right]$$

- In the region $\hat{r} \gg 1$, it can be simplified as

$$\frac{(2\pi)^3}{\beta\alpha^2 m}\tilde{G}_{Q,\parallel}(\hat{r}) \approx \frac{16\hat{z}^2 - 8|\hat{b}_\perp|^2}{|\hat{r}|^5}$$

It is a dipole shape.



- The existence of such a dipole term is not surprising
  - The monopole vanishes because there is no on-shell radioactive photon
  - While the dipole is forbidden by the spatial rotational symmetry in the static case, the relativistic motion breaks the Lorentz symmetry, which allows the existence of dipole.

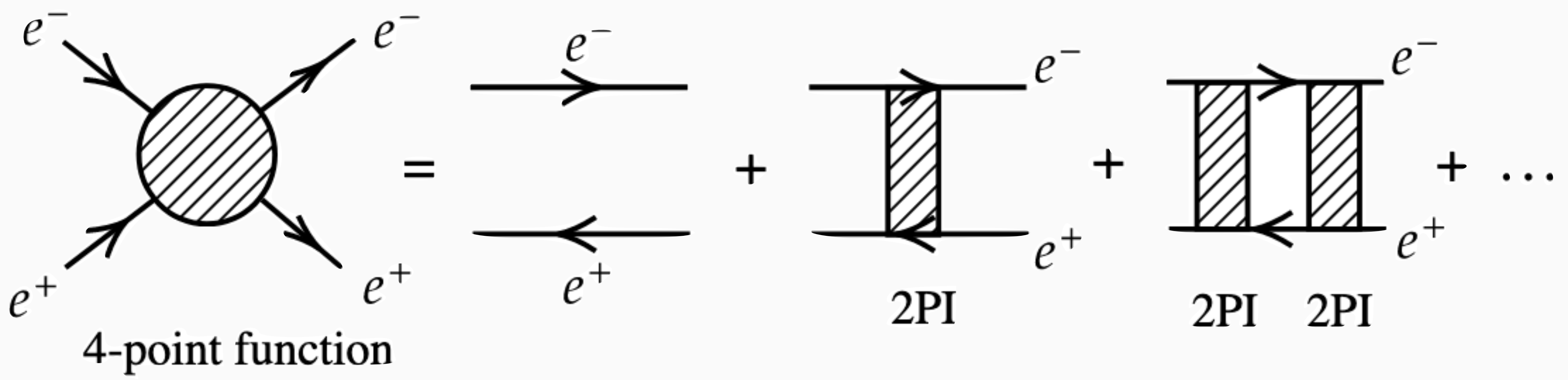
Summary

- **In this work, we computed the photon momentum distribution in relativistic positronium;**
- **The analysis is carried out using Fock state expansion and effective field theory within the framework of old-fashioned perturbation theory;**
- **In the center-of-mass frame, the photon momentum distribution vanishes at leading order. However, in the relativistic case, we find that the distribution exhibits a dipole shape in the long-range region ($\hat{r} \gg 1$) of coordinate space;**
- **The computed photon momentum distribution approaches the same ultra-relativistic limit ($\gamma \to \infty$) as found in the reference.**

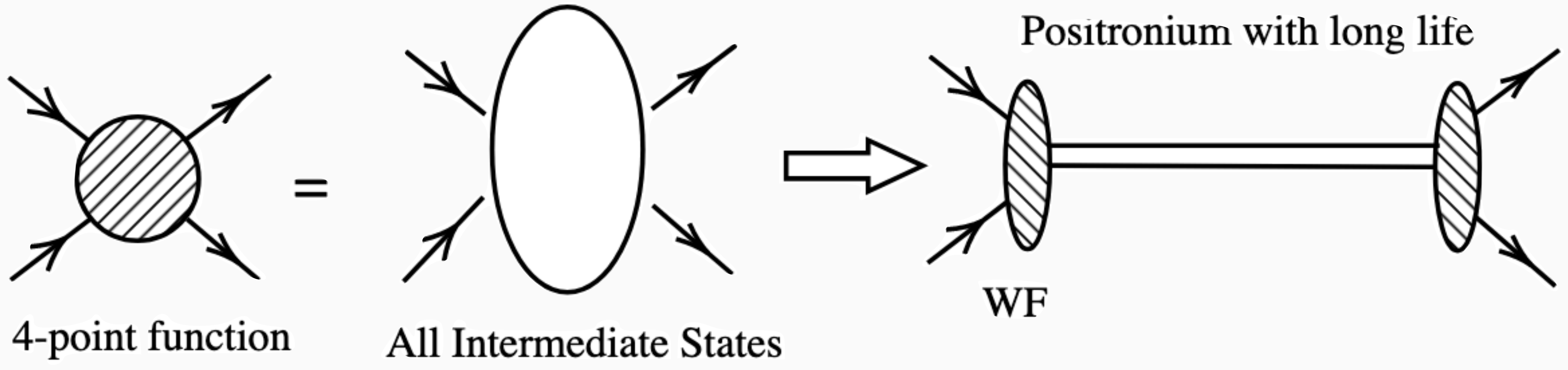
Thank you

Backup

Bethe-Salpeter Equation



- The correlation function is $G = \langle \Omega | \bar{\psi}\psi\bar{\psi}\psi | \Omega \rangle$, 2PI represents 2-point irreducible kernel K, and the free propagator is denoted as S. Then we have iterative equation $G = S_1S_2 + S_1S_2K_{12}G$.



- We can give an ansatz of the 4-point correlation function near the pole of M as $G \approx \dfrac{\bar{\varphi}_{\overrightarrow{P}}\varphi_{\overrightarrow{P}}}{P^2 - M^2}$.

- Substituting the ansatz into the iterative equation, and taking the residual at both sides, we get

$$\varphi_{\overrightarrow{P}} = S_1S_2K_{12}\varphi_{\overrightarrow{P}}$$

Reparameterization Symmetry

- **In the EFT for relativistic positronium, the fermion momentum is divided into two parts, this division introduces some ambiguities which should not affect physical results.**
- **Based on this argument, the Lagrangian should remain invariant under the reparameterization transformation**

$$p = mv + q \to p = m\left(v + \frac{l}{m}\right) + (q - l)$$

- **Take the scalar field as an example, under the reparameterization transformation, we have**

$$\phi_v^*[f(v, iD)]\phi_v \to \phi_w^*[f(v, iD + l)]\phi_w \text{ , in which } \phi_w = e^{il\cdot x}\phi_v \text{ and } w = v + l/m$$

- **If we rename the variable $w \to v$, the RHS above becomes**

$$\phi_v^*[f(v - l/m, iD + l)]\phi_v$$

- **To keep the Lagrangian term invariant, we can construct the invariant velocity as $V^\mu = v^\mu + \frac{iD^\mu}{m}$, which is the only quantity which respects reparameterization symmetry at order $\mathcal{O}(\alpha)$.**
- **To construct a Lorentz scalar in terms of the invariant velocity, we have**

$$(mV)^2 = m^2 + 2m(iD \cdot v) + (iD)^2 = m^2 + 2m\left((iD \cdot v) + \frac{(iD)^2}{2m}\right)$$

- **The reparameterization symmetry can be understood as the residual effect of the Lorentz symmetry.**

Spin Wave Function

Finally, Eq. B15 is verified with the normalization factor $N(\vec{P}) = \sqrt{\gamma} N(\vec{0})$. Recalling the leading-order term in the Fock space expansion of the positronium in Eq. 17, the spin wave function $S^s_{s_1,s_2}(\vec{P})$ in Eq. 23 can be defined as

$$S^{s=0}_{s_1=h_1,s_2=h_2}(\vec{P}) \equiv \frac{\gamma}{\sqrt{2(2E_{\vec{p}})(2E_{\vec{P}-\vec{p}})}} \bar{u}(h_1,\vec{p})\gamma^5 v(h_2,\vec{P}-\vec{p}) \ , \tag{B21}$$

where γ is the normalization factor. This is the spin wave function of the pseudo-scalar positronium in the relativistic motion. For positronium with higher spin states, things can be more complicated. Specifically, the spin-1 state corresponds to the Proca equation [42], while the spin-2 state is associated with the Fierz-Pauli equation [43]. It is important to note that the dynamical effects of the Poincare transformation have not been considered in this section. This omission is justified by the fact that, for the moving positronium case, these dynamical effects are irrelevant to the spin wave function at leading order. Consequently, the dynamical effects are addressed in the calculation of the spatial wave function in Sect. IV.

In the center-of-mass frame, the spin wave function becomes

$$S^{s=0}_{s_1=h_1,s_2=h_2}(\vec{P}=0) = \frac{1}{2\sqrt{2}E_{\vec{k}}}\bar{u}(h_1,\vec{k})\gamma^5 v(h_2,-\vec{k}) = \frac{1}{2\sqrt{2}E_{\vec{k}}}\xi^\dagger_{h_1}\eta_{h_2}(-k\cdot\sigma - k\cdot\bar{\sigma}) \approx -\frac{\sqrt{2}}{2}\delta_{h_1,h_2} \ . \tag{B22}$$

In the boosted frame where the Lorentz factor satisfies $\gamma \gg 1$, we have the approximation $\vec{p} \approx \vec{P}/2$, the spin wave function becomes

$$S^{s=0}_{s_1=h_1,s_2=h_2}(\vec{P}) = \frac{\gamma}{\sqrt{2(2E_{\vec{p}})(2E_{\vec{P}-\vec{p}})}}\bar{u}(h_1,\vec{p})\gamma^5 v(h_2,\vec{P}-\vec{p}) \approx \frac{\gamma}{2\sqrt{2}E_{\vec{p}}}\xi^\dagger_{h_1}\eta_{h_2}(-m-m) = -\frac{\sqrt{2}}{2}\delta_{h_1,h_2} \ . \tag{B23}$$

Therefore, we have the normalization condition of the spin wave function as

$$\sum_{s_1,s_2} |S^{s=0}_{s_1=h_1,s_2=h_2}(\vec{P})|^2 = 1 \ . \tag{B24}$$

Connection to TMDs

- We can change another point of view to understand the photon momentum operator.

$$\tilde{A}(-\vec{k})\tilde{A}(\vec{k}) = \int d^3x_1 \int d^3x_2\ A(\vec{x}_1)e^{i\vec{k}\cdot\vec{x}_1}A(\vec{x}_2)e^{-i\vec{k}\cdot\vec{x}_2} = \int d^3X \int d^3x\ A\left(\vec{X}+\frac{\vec{x}}{2}\right)A\left(\vec{X}-\frac{\vec{x}}{2}\right)e^{i\vec{k}\cdot\vec{x}},$$

where $\vec{X} = (\vec{x}_1 + \vec{x}_2)/2$ and $\vec{x} = (\vec{x}_1 - \vec{x}_2)$.

- In comparison, the Wigner function is defined as

$$W\left(\vec{R},\vec{k}\right) = \int d^3\Delta\vec{r}\ \psi^*\left(\vec{R}+\frac{\Delta\vec{r}}{2}\right)\psi\left(\vec{R}-\frac{\Delta\vec{r}}{2}\right)e^{i\vec{k}\cdot\Delta\vec{r}}$$

- The TMDs, which is the distribution function in momentum space, are defined as

$$n\left(\vec{k}\right) \equiv \int d^3\vec{R}\ W\left(\vec{R},\vec{k}\right)$$

- It is obvious that the photon momentum operator has a form similar to that of the TMD distributions that describe the hadron inner structure. Although QCD bound states cannot be perturbatively expanded as positronium because of the non-perturbative scale Λ_{QCD}, the connections between this paper and TMD physics are still worth investigating in the future.