

3D Imaging of the Pion on a Fine Lattice

Phys. Rev. D 112 (2025) 2504.04625

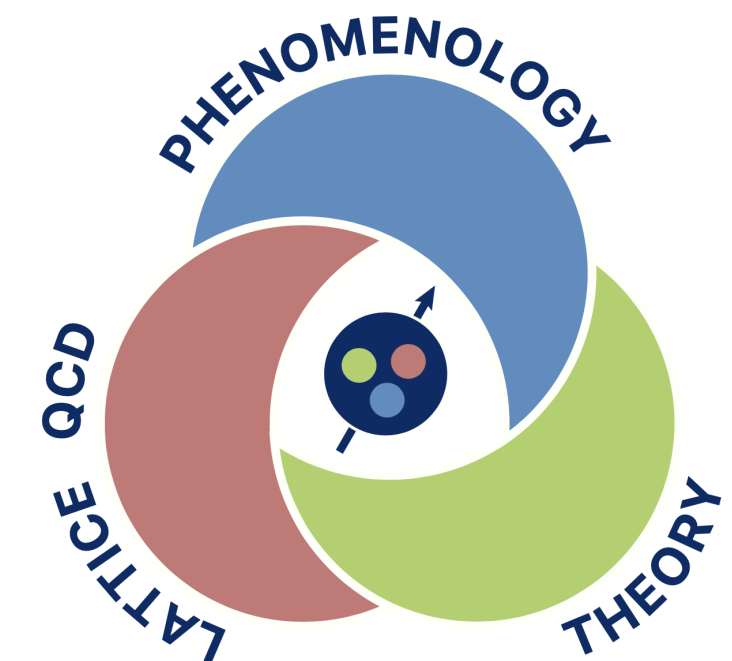
And on going projects ...

Jinchen He

In Collaboration with D. Bollweg, X. Gao, S. Mukherjee and Y. Zhao

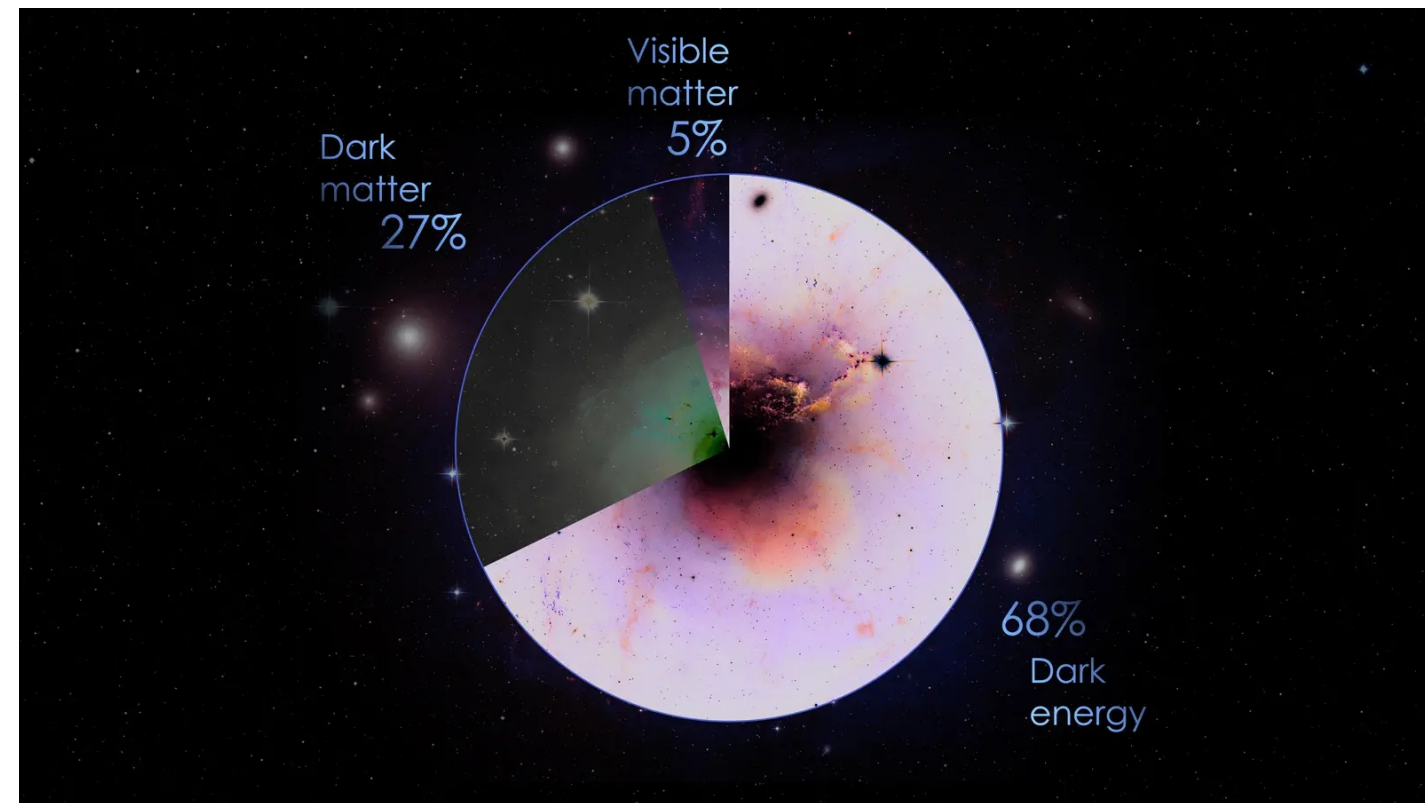
Student Lunch Talk

2026/4/15



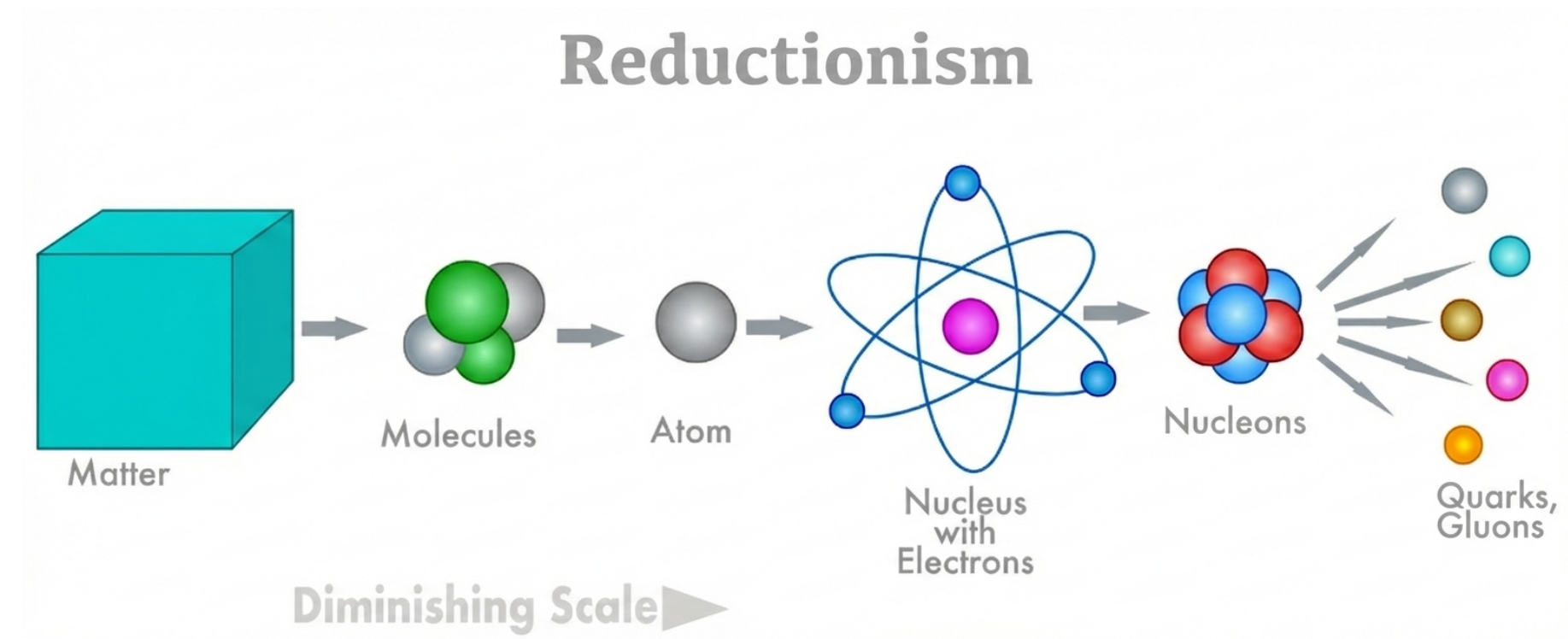
Visible Universe

- Only 5% of the universe is visible; *D. N. Spergel, Science 347.6226 (2015)*
- The visible universe can be understood as a hierarchy: molecules → atoms → nuclei → nucleons → ...;

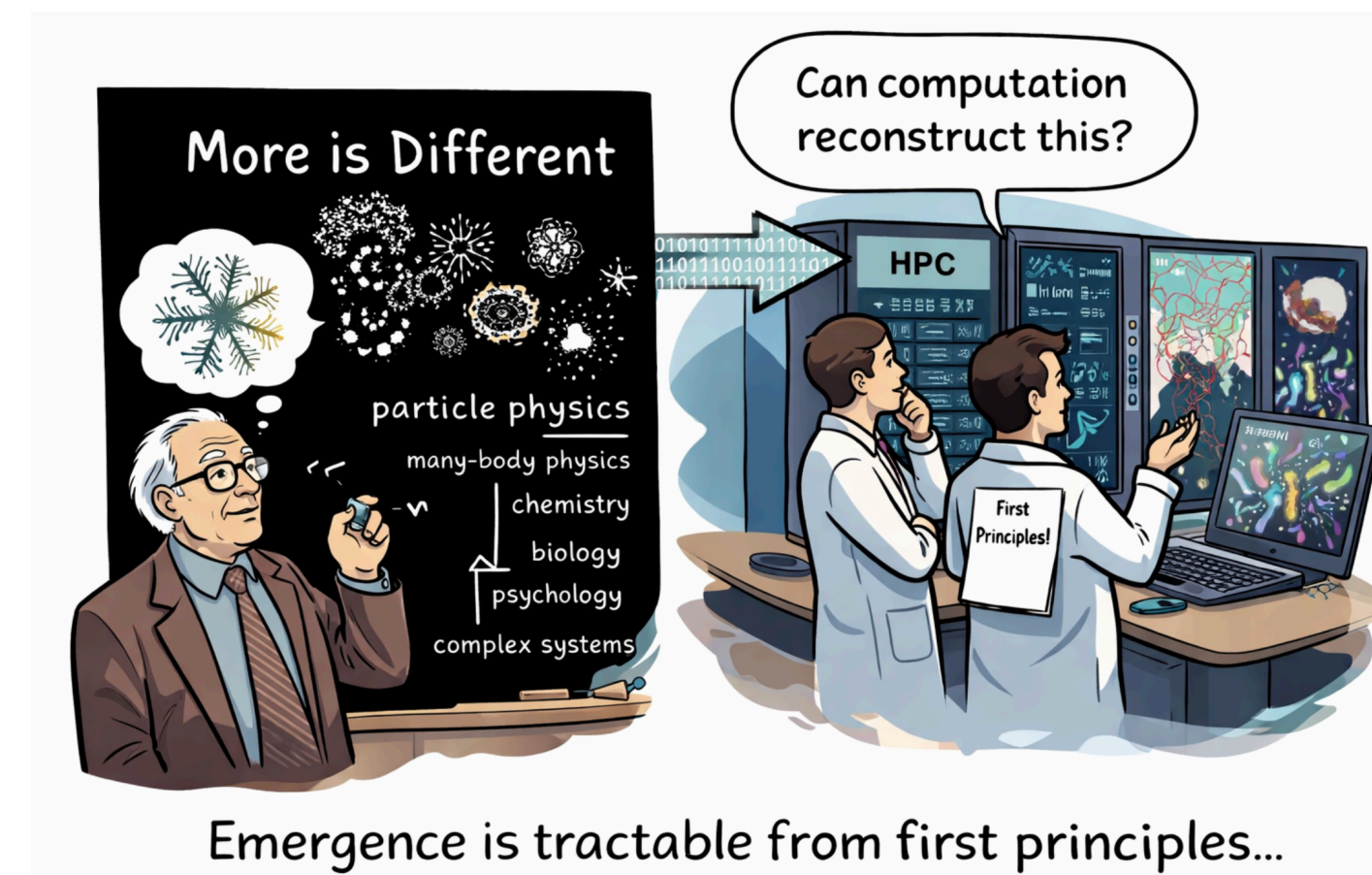


Cr. NASA's Goddard Space Flight Center

Reductionism motivates the search for fundamental degrees of freedom.



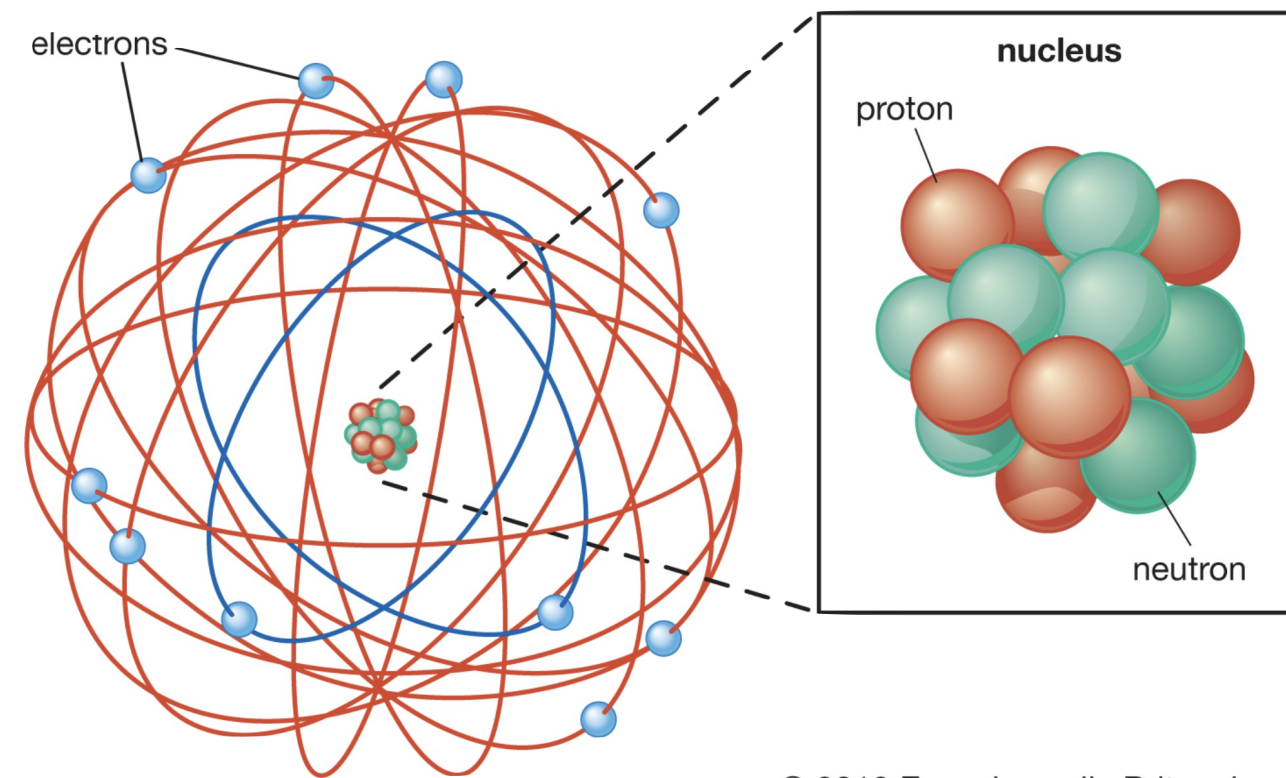
- More is different? *P. W. Anderson, Science 177.4047 (1972)*
- Emergence *After 50 years, things are changing...*
- Broken symmetry *Computation becomes more and more important in this era*
- Complexity (large number of d.o.f., strongly coupled system)



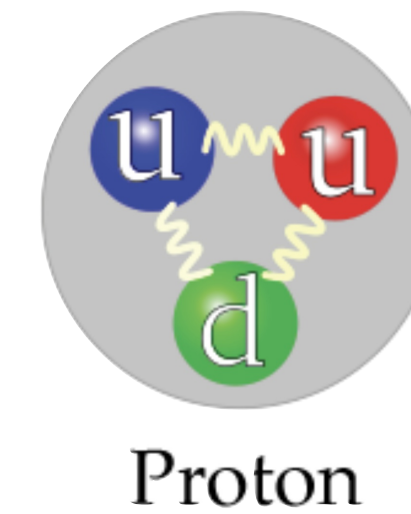
- **Modern computation makes macroscopic emergence increasingly tractable from first principles.**

Visible Universe

- The inner structure of hadrons are still sophisticated if we step closer;

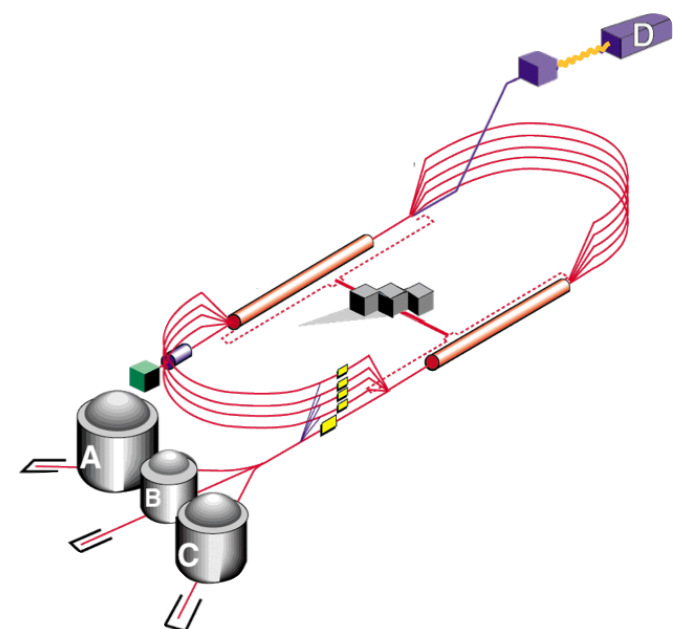


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- Many experiments have been designed to probe the internal structure of hadrons.

CEBAF(JLab)



Cr. Dave Gaskell

HERA



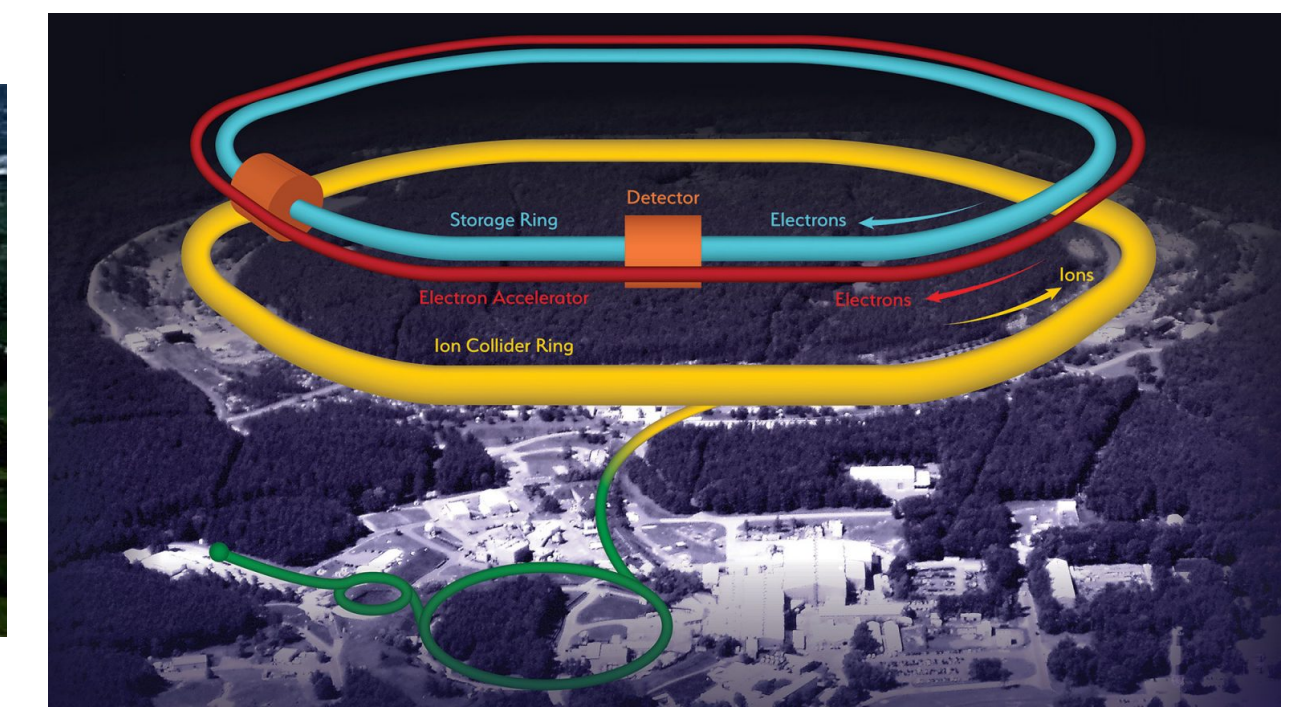
Cr. DESY

LHC



Cr. CERN

EIC



Cr. BNL

Parton Physics

- Our knowledge of hadrons is still limited:

- Spin, mass ...

Perturbation theory

Non-relativistic EFTs

- How to describe a strongly coupled relativistic moving bound state?

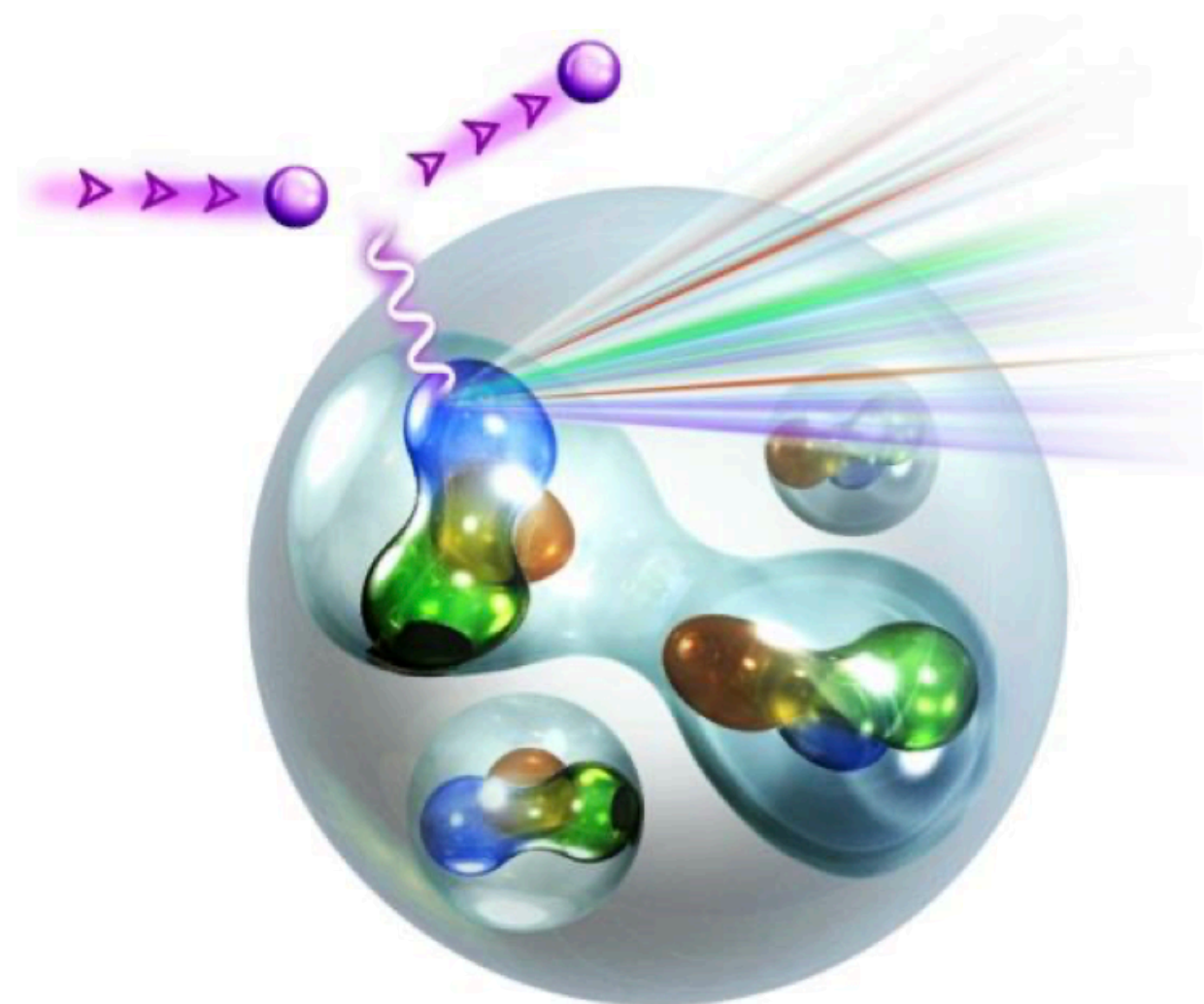
- The language from Feynman: Parton Model in the infinite momentum frame

R. P. Feynman, Conf. Proc. C 690905 (1969)

- Quarks and gluons (partons) are “frozen” in the transverse plane;

- During a high-energy collision, the struck parton appears like a free particle.

Bjorken Scaling → point-like structure



Cr. Dave Gaskell

The many faces of the proton

QCD bound state of **quarks** and **gluons**

Origin of mass?

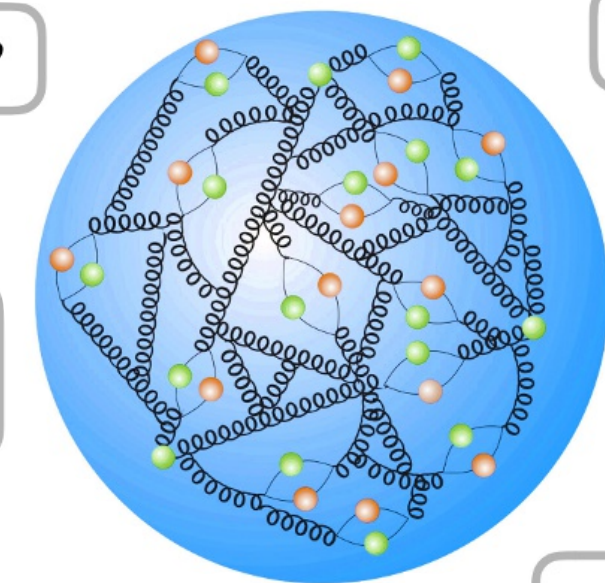
Origin of spin?

Gluon-dominated matter?

3D imaging?

Heavy quark content?

Nuclear modifications?



Cr. Juan Rojo

Collinear Factorization

$$\sigma_{\text{DIS}} \propto \left| \begin{array}{c} l \\ l' \\ q \\ P \end{array} \right|^2 \approx \left| \begin{array}{c} k \approx xP \\ P \end{array} \right|^2 \otimes \left| \begin{array}{c} l \\ l' \\ q \\ xP \end{array} \right|^2$$

The diagram shows the collinear factorization of the deep inelastic scattering cross-section. It is expressed as the product of three terms: the squared amplitude of the hard scattering process (involving lepton lines l, l' and a quark line q), the squared parton distribution function (PDF) (involving the proton momentum P and the parton momentum $k \approx xP$), and the squared amplitude of the parton distribution function (involving the lepton lines l, l' and the parton momentum xP).

R. Boussarie, et al., "TMD Handbook", [arXiv:2304.03302 [hep-ph]]

3D Imaging of Hadron

x is the momentum fraction in the longitudinal (hadron momentum) direction.

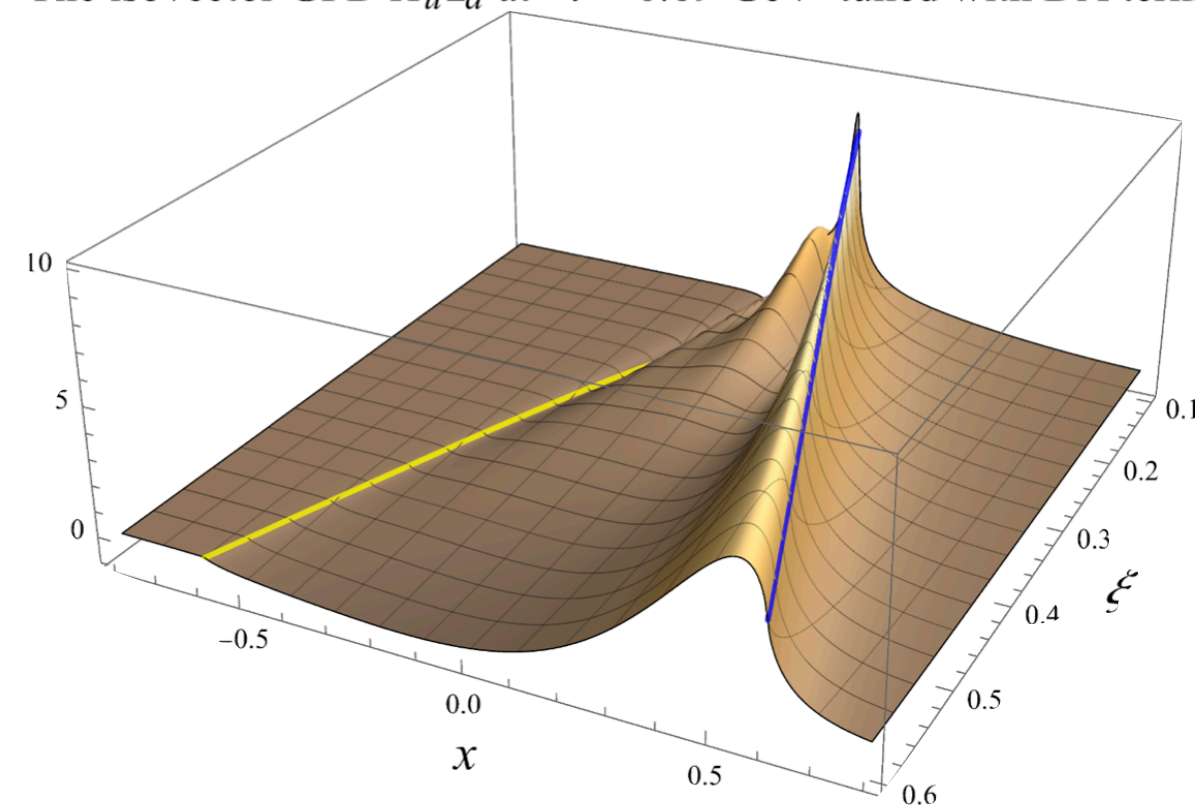
Wigner Distribution / GTMD

$$W(x, \vec{r}_\perp, \vec{k}_\perp)$$

$$\int d^2\vec{k}_\perp$$

Generalized Parton Distributions (GPDs)

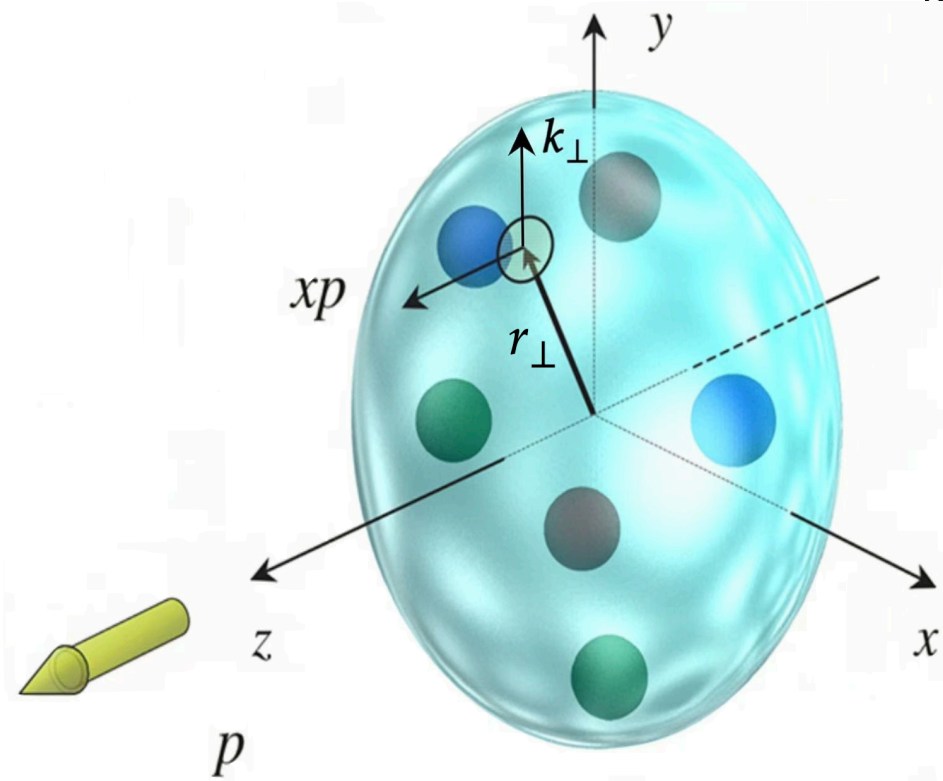
The isovector GPD H_{u-d} at $-t = 0.69 \text{ GeV}^2$ tuned with DA terms



Y. Guo, et al., JHEP 09 (2022)

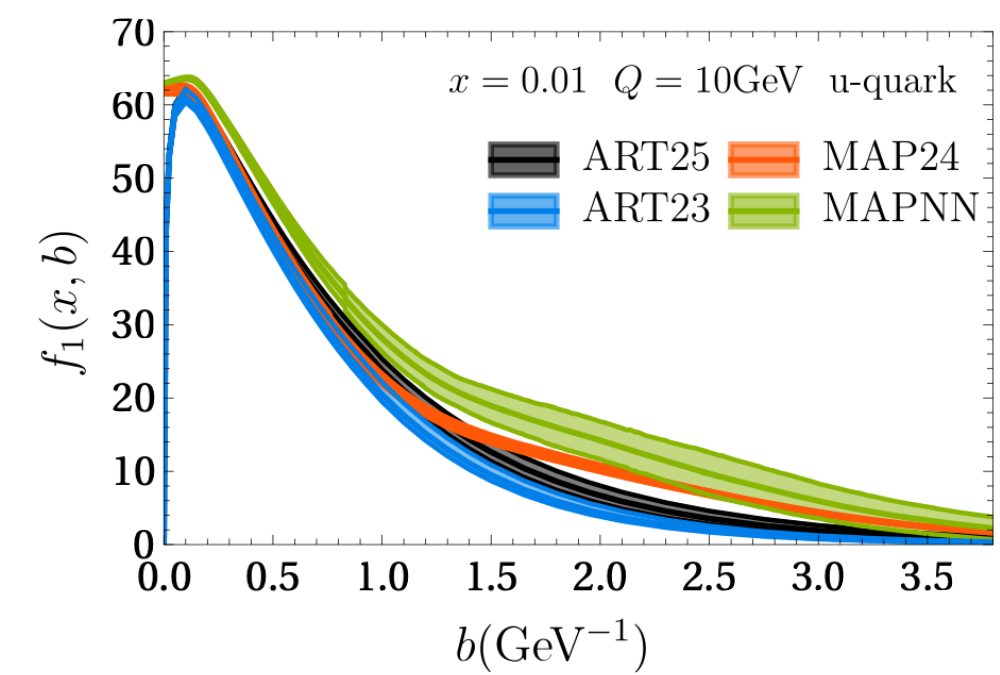
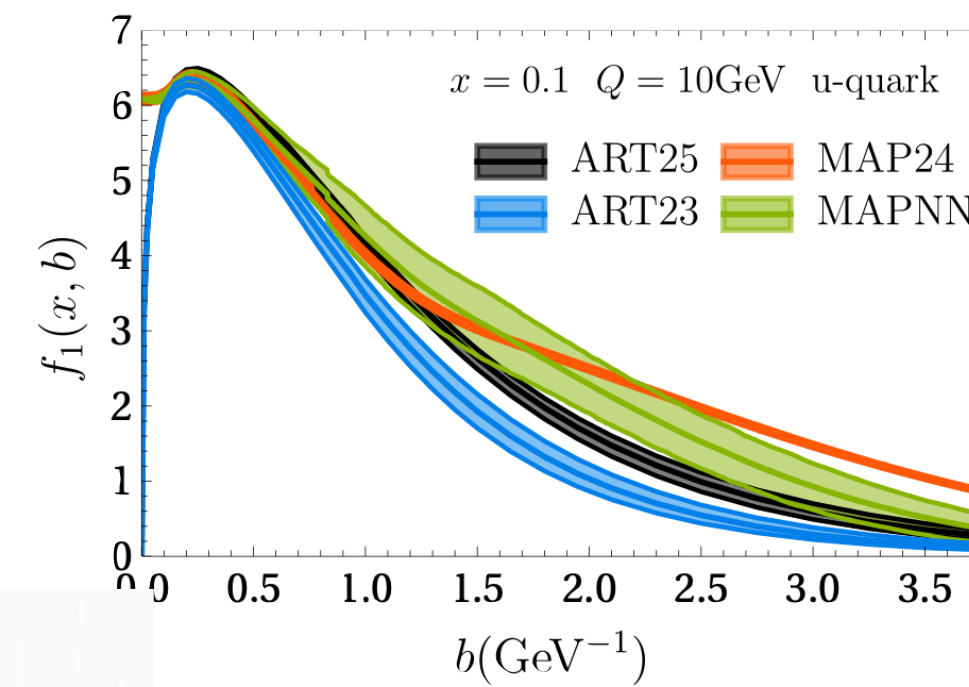
Latest results of GUMP 1.0 in *Y. Guo, et al., PRL 135 (2025)*

$$\int d^2\vec{r}_\perp$$



$$\int d^2\vec{r}_\perp$$

Transverse-Momentum-Dependent distributions (TMDs)

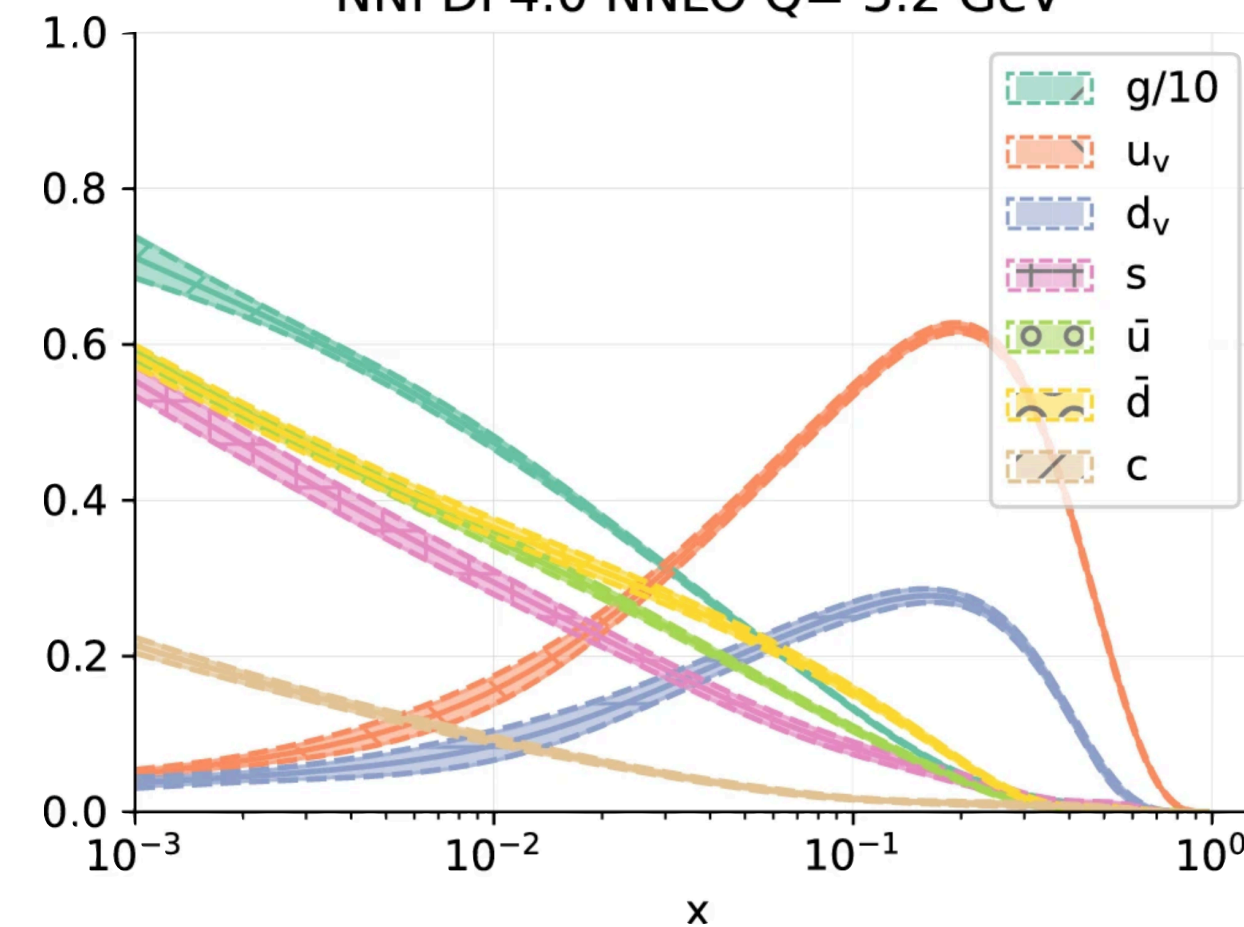


V. Moos, et al., JHEP 11 (2025)

$$\int d^2\vec{k}_\perp$$

Parton Distribution Functions (PDFs)

NNPDF4.0 NNLO $Q = 3.2 \text{ GeV}$



R. D. Ball, et al. [NNPDF], Eur. Phys. J. C 82 (2022)

TMDs in Experiments

- **TMDPDFs**: the distribution densities of finding a parton carrying a longitudinal momentum fraction x and transverse momentum $k_\perp$ in a hadron;

TMD Factorization

$$\sigma_{\text{DY}} \propto \left| \text{Diagram 1} \right|^2 \approx \left| \text{Diagram 2} \right|^2 \otimes \left| \text{Diagram 3} \right|^2 \otimes \left| \text{Diagram 4} \right|^2$$

But why we need TMD factorization?

- In the Parton Model

When $q_T \sim Q$, collinear factorization is enough

$$\frac{d\sigma}{d^4q} = \sum_{i,j} \int_{x_a}^1 d\xi_a \int_{x_b}^1 d\xi_b \boxed{f_{i/H_a}(\xi_a)} f_{j/H_b}(\xi_b) \frac{d\hat{\sigma}_{ij}(\xi_a, \xi_b)}{d^4q} \left[1 + O\left(\frac{\Lambda_{\text{QCD}}^2}{q_T^2}, \frac{\Lambda_{\text{QCD}}^2}{Q^2}\right) \right]$$

When $\Lambda_{\text{QCD}} \lesssim q_T \ll Q$, TMD factorization is needed

$$\frac{d\sigma}{d^4q} = \frac{1}{s} \sum_{i \in \text{flavors}} \hat{\sigma}_i^{\text{TMD}}(Q) \int d^2\mathbf{k}_T \boxed{f_{i/H_a}(x_a, \mathbf{k}_T)} f_{\bar{i}/H_b}(x_b, \mathbf{q}_T - \mathbf{k}_T) \left[1 + O\left(\frac{q_T^2}{Q^2}, \frac{\Lambda_{\text{QCD}}^2}{Q^2}\right) \right]$$

{ Soft mode
Scale separation
Large log: $\ln(Q/q_T)$

Phenomenological Extraction of TMDs

- Significant progress has been made in the phenomenological parameterizations of TMDs

Some of groups will incorporate neural network;

Some of groups will include lattice data;

- Collins-Soper kernel (CS kernel): rapidity evolution kernel of TMDs

A. Bacchetta, et al. (MAP) JHEP 08 (2024); V. Moos, et al., JHEP 11 (2025) ...

- Nucleon TMDs

- Unpolarized *M. Bury, et al., JHEP 10 (2022); A. Bacchetta, et al., JHEP 10 (2022); V. Moos, et al., JHEP 05 (2024) ...*

- Sivers *M. Bury, et al., PRL 126 (2021); I. P. Fernando, et al., Phys.Rev.D 108 (2023) ...*

- Boer-Mulders *Z. Lu, et al., Phys.Rev.D 81 (2010); X. Liu, et al., Eur.Phys.J.C 81 (2021) ...*

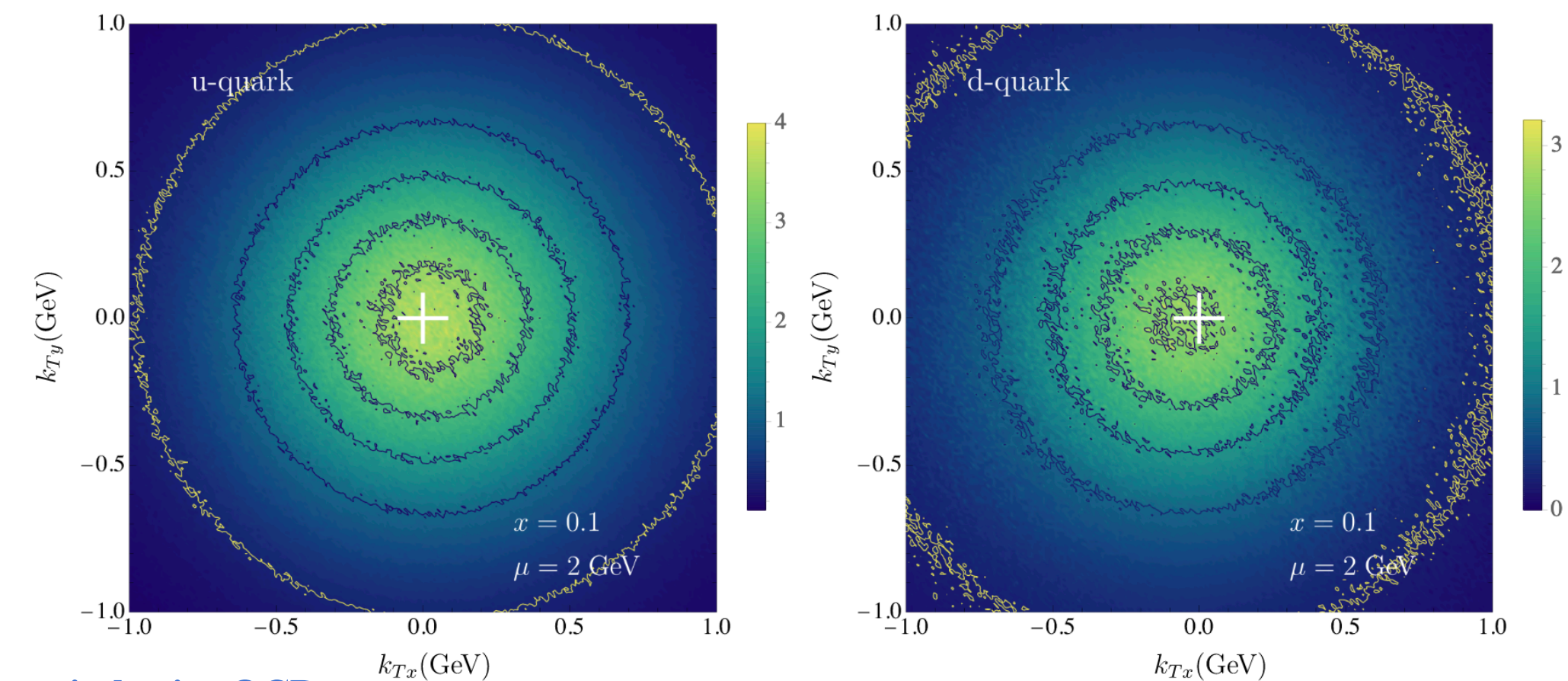
- **Pion TMDs: much less is known about the TMDs of the pion**

Due to the limited experimental data, but pion is relatively easy in lattice QCD

- Unpolarized *A. Vladimirov, JHEP 10 (2019); M. Cerutti, et al. (MAP), Phys.Rev.D 107 (2023); P. C. Barry, et al. (JAM) Phys.Rev.D 108 (2023)*

M. Bury, et al., JHEP 05 (2021)

Tomographic scan of the nucleon via quark density function



As the lightest pseudo Nambu-Goldstone boson, the 3D structure of pion will help us understand the strong interaction, such as the origin of chiral-symmetry breaking.

Lattice QCD: First-principle Non-perturbative Method

- Path integral formalism

$$Z = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS_{\text{QCD}}[A,\psi,\bar{\psi}]} \xrightarrow[\text{Wick rotation}]{\substack{\text{Wilson link} \\ U_\mu(x) = e^{igaA^\mu(x)} \\ t \rightarrow -it_E}} Z_E = \int \mathcal{D}U e^{-S_E^g[U]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\bar{\psi} M[U] \psi} = \int \mathcal{D}U e^{-S_E^g[U]} \det M[U]$$

- If we define the probability distribution of gauge configuration: $P[U] = \frac{1}{Z_E} e^{-S_E^g[U]} \det M[U]$ Importance Sampling

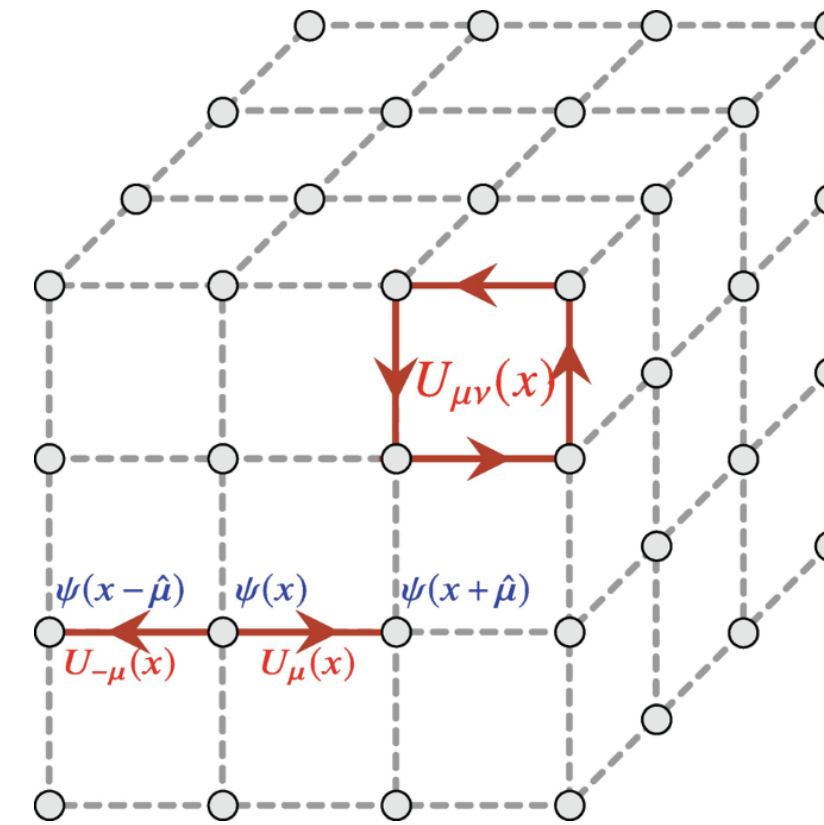
- Monte Carlo sampling

$$\langle \hat{O} \rangle = \frac{1}{Z_E} \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} \hat{O} e^{-S_{\text{QCD}}[A,\psi,\bar{\psi}]} = \frac{1}{Z_E} \int \mathcal{D}U e^{-S_E^g[U]} \det M[U] O[U] = \int \mathcal{D}U P[U] O[U] = \frac{1}{N} \sum_{i=1}^N O[U^{(i)}]$$

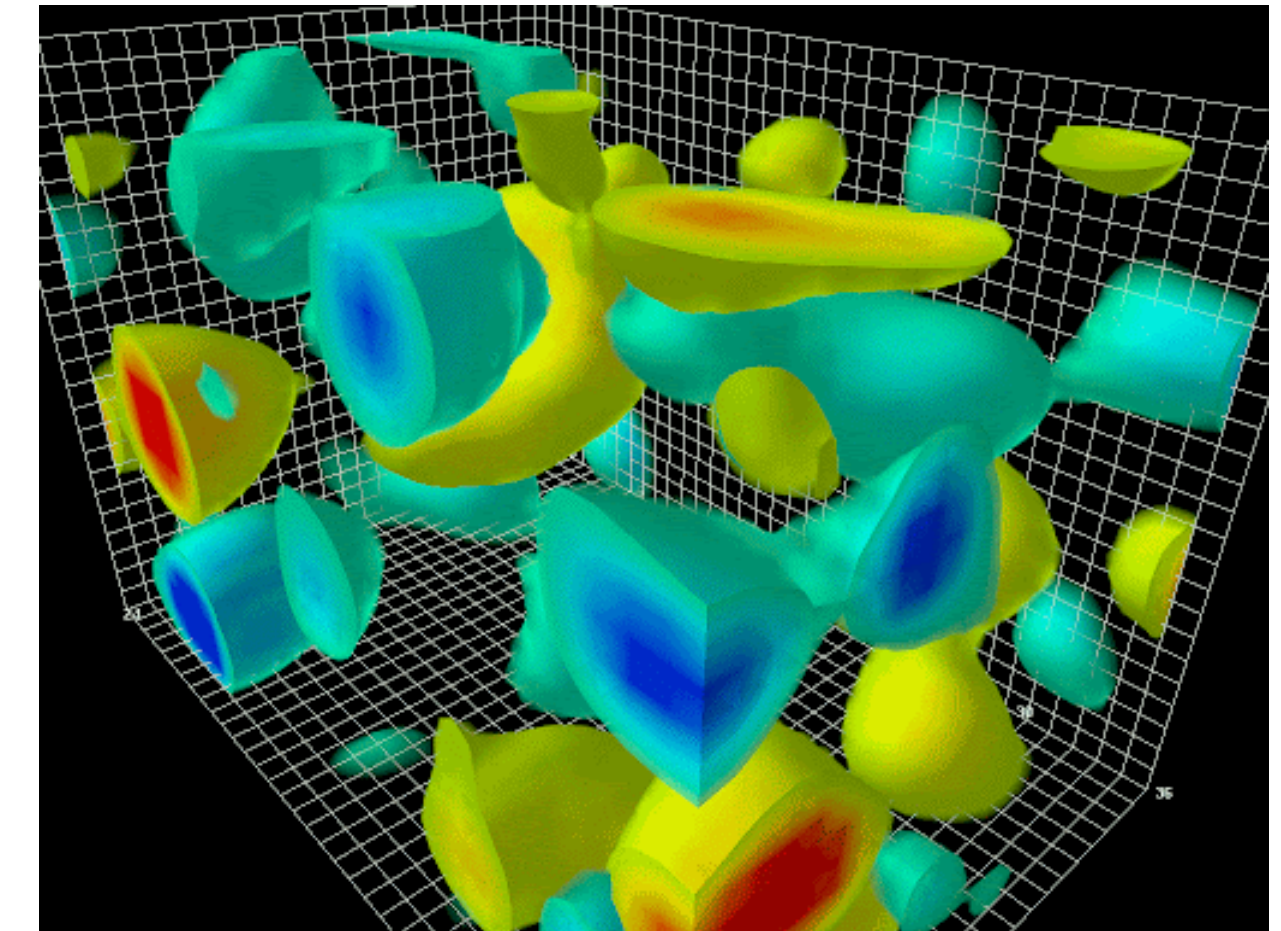
- Once we get an ensemble with the probability distribution above, the path integral can be replaced by sample average;
- Importance sampling ensures that typical configurations dominate, so a finite ensemble suffices to approximate the path integral.

Lattice QCD: A Wilsonian View of QCD

- Lattice QCD as an effective theory
 - A complete theory or an effective theory?
 - Natural regulator: finite lattice spacing, finite volume
- Wilson's renormalization group perspective
 - Physics is organized by scale: integrate out short-distance degrees of freedom
 - Continuum QCD is defined along a renormalized trajectory controlled by a UV fixed point
- Lattice QCD provides a nonperturbative and well-defined regularization from which continuum QCD can be constructed
 - From path integral to finite dimensional integral, with a well-defined measure
 - We define QCD as a Euclidean statistical theory, and real-time physics obtained through analytic continuation



Cr. Olaf Kaczmarek



Cr. Derek Leinweber

Lattice QCD Calculation of TMDs

- As a first-principles non-perturbative method, Lattice QCD provides independent predictions of TMDs.

- Mellin Moments *B. Yoon, et al., 1601.05717; B. Yoon, et al., Phys. Rev. D 96 (2017) ...*

- Large Momentum Effective Theory (LaMET)

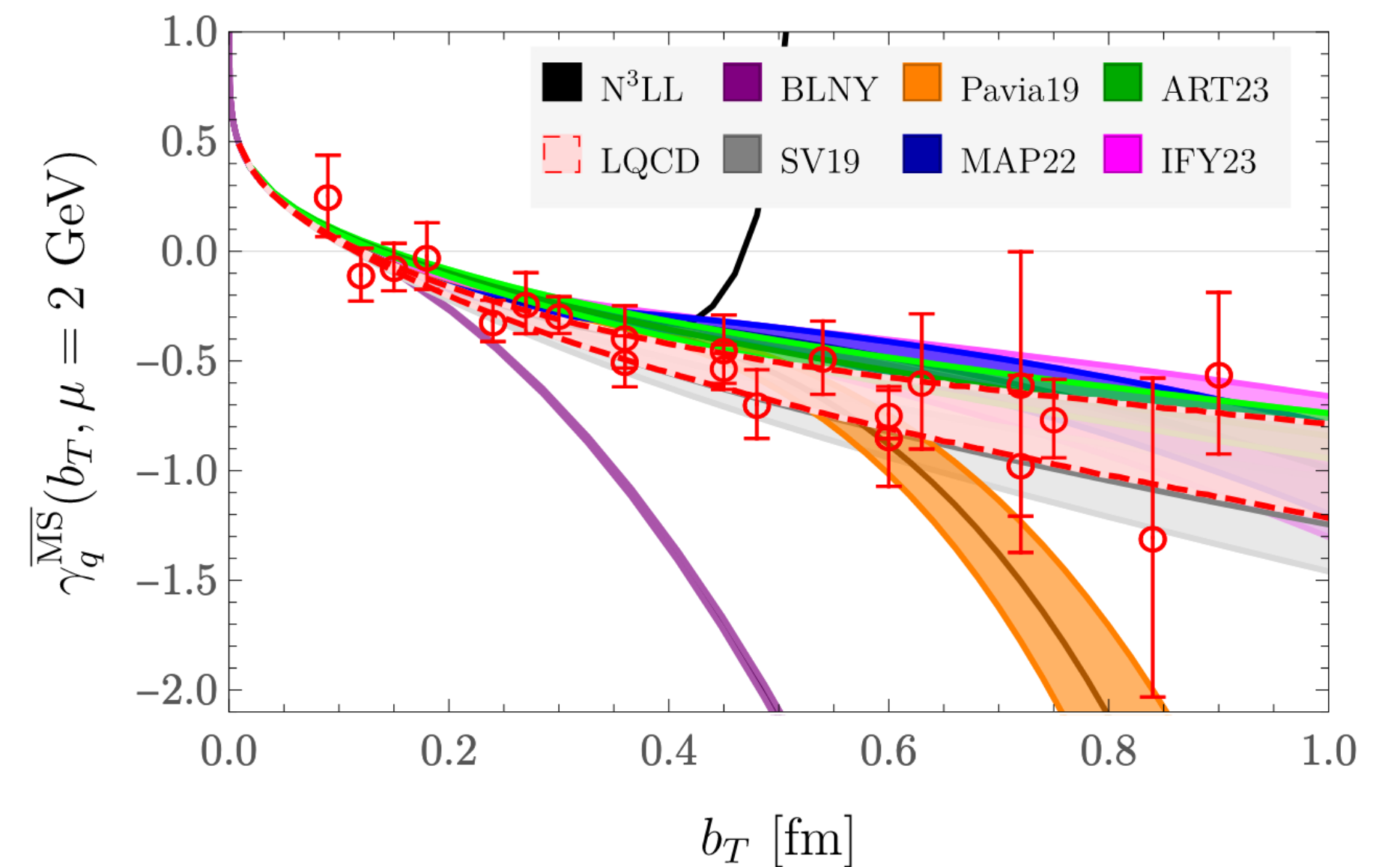
- CS kernel *M. H. Chu, et al. (LPC), JHEP 08 (2023); A. Avkhadiev et al., PRL 132 (2024); D. Bollweg, et al., Phys. Lett. B 852 (2024); J-X Tan, et al. (LPC), 2511.22547; A. Avkhadiev et al., 2510.26489 ...*

- Intrinsic soft function *Q. A. Zhang, et al. (LPC), PRL 125 (2020); M. H. Chu, et al. (LPC), JHEP 08 (2023)*

- Unpolarized *JH, et al. (LPC), Phys.Rev.D 109 (2024)*

- Helicity *D. Bollweg, X. Gao, S. Mukherjee and Y. Zhao, PRL 135 (2025)*

- Boer-Mulders *L. Walter, et al. (LPC), PRD 111 (2025); L. Ma, et al. (LPC), JHEP 08 (2025)*



A. Avkhadiev et al., PRL 132 (2024)

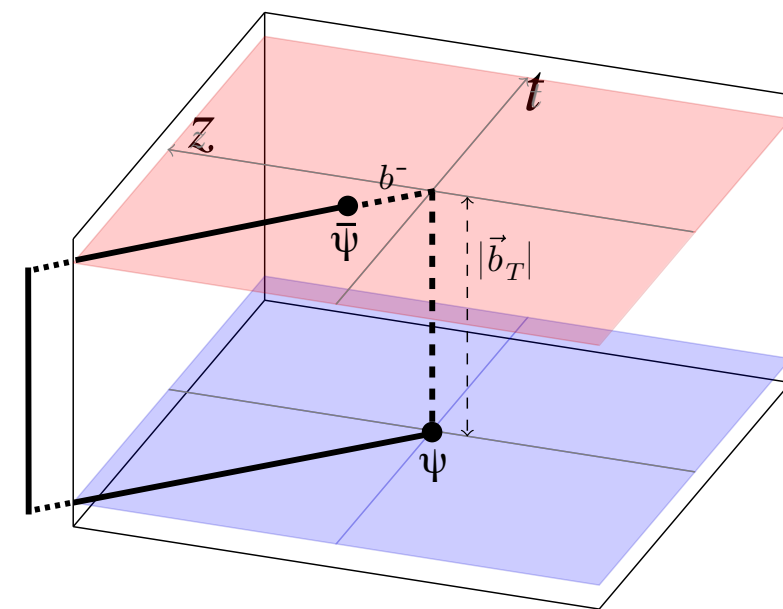
Large-Momentum Effective Theory (LaMET)

- TMDPDF is defined from a light-cone correlator in a hadron, which is boost invariant.

$$f(x, b_{\perp}, \dots) = \int_{-\infty}^{\infty} \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \left\langle P \left| \bar{\psi}(b^{\mu}) W_{\square}(b^{\mu}, 0) \frac{\gamma^+}{2} \psi(0) \right| P \right\rangle \xleftrightarrow{\text{Parton model}} \left\langle |\vec{P}| = \infty \left| O(t=0) \right| |\vec{P}| = \infty \right\rangle$$

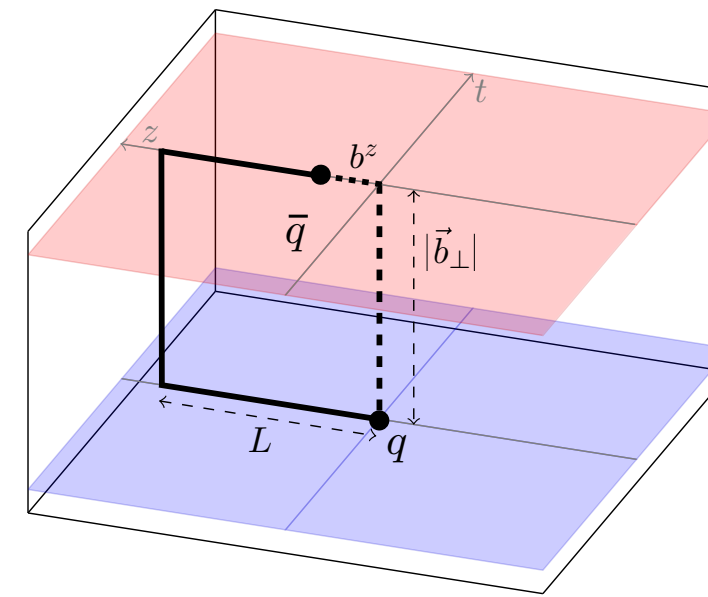
- Define a quasi distribution with large-momentum states and time-independent operators.

$$\tilde{f}_{\Gamma}^0(x, b_{\perp}, P^z, \mu) = P^z \int \frac{dz}{2\pi} e^{iz(xP^z)} \frac{1}{2P^t} \langle P | \bar{\psi}_0(z, b_{\perp}) W_{\square}(z, b_{\perp}; 0) \Gamma \psi_0(0) | P \rangle, \quad \Lambda_{\text{QCD}} \ll |\vec{P}| \ll \frac{\pi}{a}$$



X. Ji, Phys.Rev.Lett. 110 (2013)
X. Ji, et al., Rev.Mod.Phys. 93 (2021)
X. Ji, Nucl. Phys. B 1007 (2024)

Lorentz boost & Matching



Different orders of limit, but pert.

Large-momentum expansion

Light-cone distribution:

Critical point with infinite correlation length
Cannot be directly calculated on the lattice

Quasi distribution:

Directly calculable on the lattice

$H_f(x, P^z; \mu) = |C_{\text{TMD}}(xP^z; \mu)|^2$ is the TMD hard kernel for matching.

- LaMET enables us to obtain the precision-controlled x-distribution of TMDs in $x \in [x_{\min}, x_{\max}]$.

$$\sqrt{S_I(b_{\perp}; \mu)} \tilde{f}_{\Gamma}(x, b_{\perp}, P^z; \mu) = f(x, b_{\perp}; \mu, \zeta) H_f(x, P^z; \mu) \exp \left[\frac{1}{2} \ln \frac{(2xP^z)^2}{\zeta} \gamma^{\overline{\text{MS}}}(b_{\perp}; \mu) \right] + \text{Power corrections}$$

Collins-Soper scale: $\zeta \sim 2(xP^+)^2$

Coulomb Gauge Method

- Define a quasi distribution in CG without Wilson line: *X. Gao, W. Y. Liu and Y. Zhao, PRD 109 (2024)*

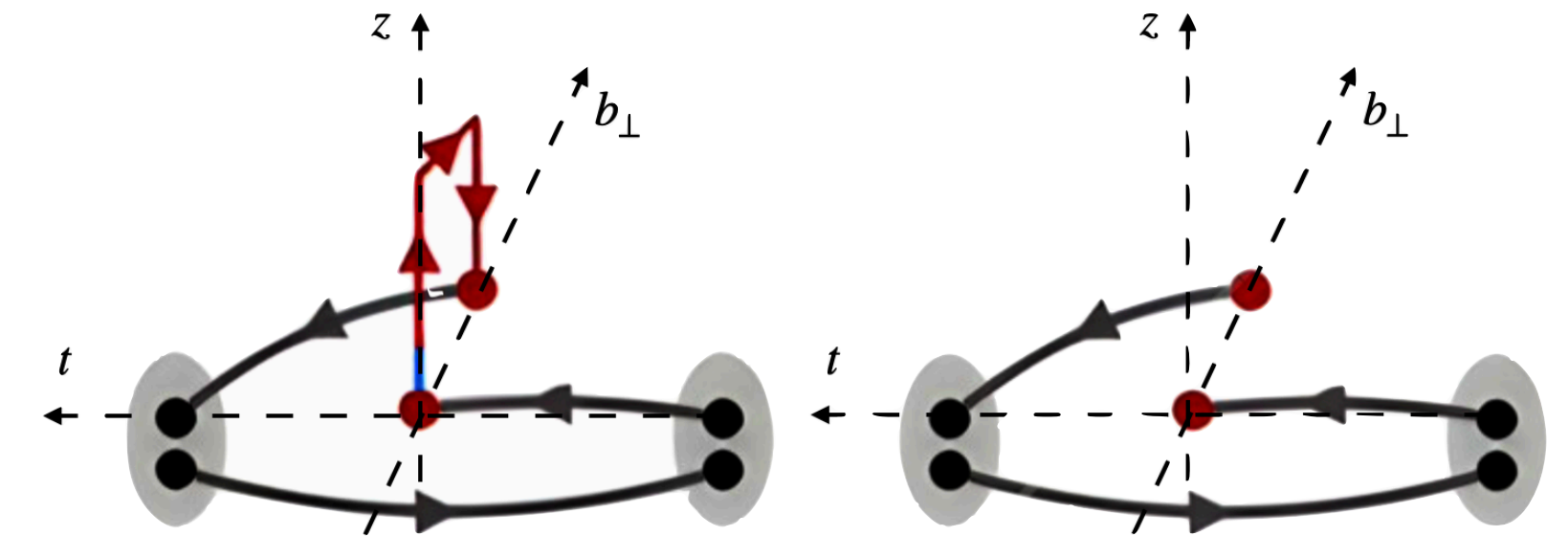
Y. Zhao, PRL 133 (2024)

$$\tilde{f}_{\text{CG}}^0(x, b_{\perp}, P^z, \mu) = P^z \int \frac{dz}{2\pi} e^{iz(xP^z)} \frac{1}{2P^t} \langle P | \bar{\psi}_0(z, b_{\perp}) \Gamma \psi_0(0) | P \rangle \Big|_{\vec{\nabla} \cdot \vec{A} = 0}$$

- Why choose CG?

X. Ji, Y. S. Liu, Y. Liu, J. H. Zhang and Y. Zhao, RMP 93 (2021)

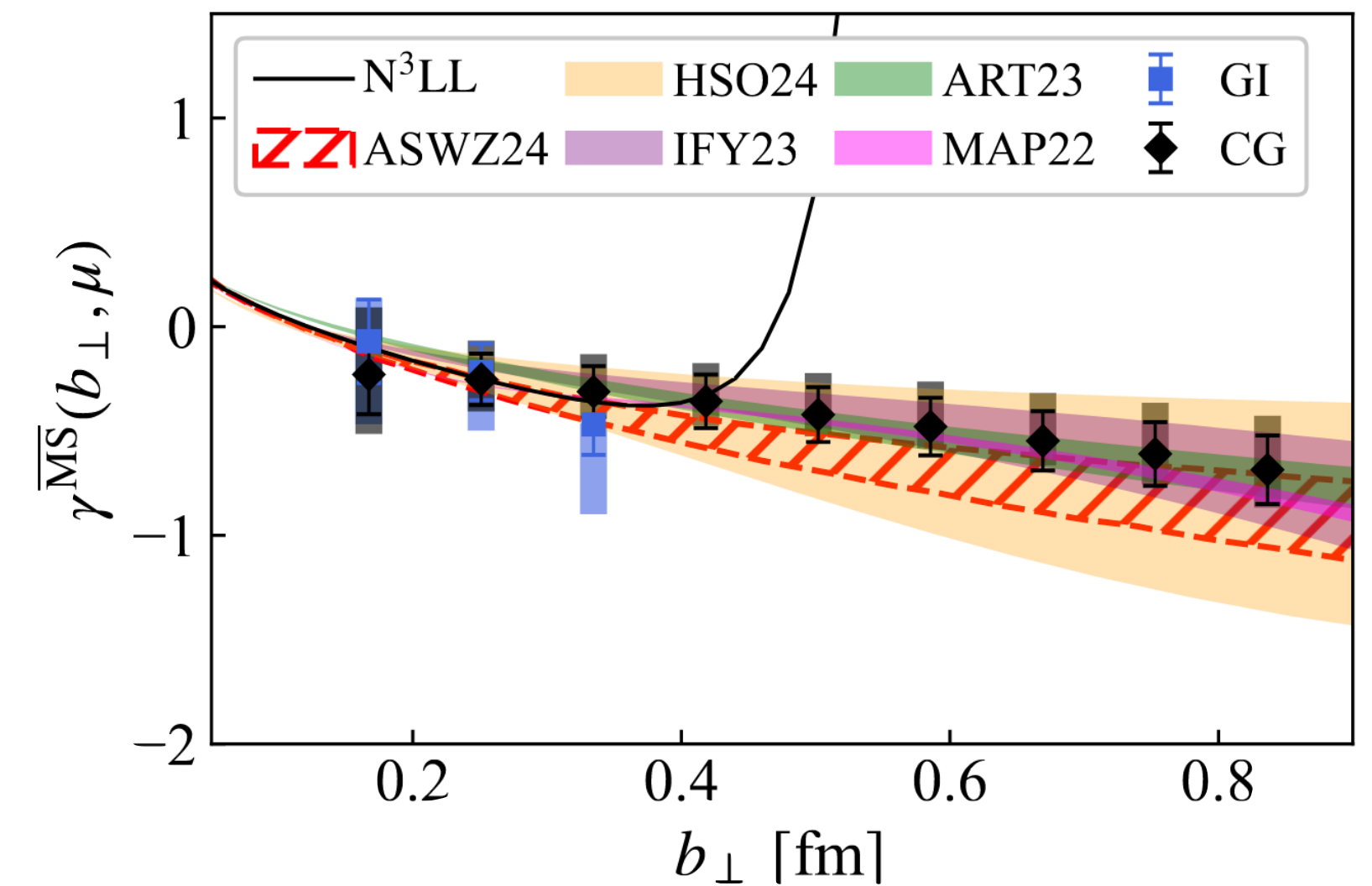
- $\vec{\nabla} \cdot \vec{A} = 0$ becomes $A^+ = 0$ in the infinite boost, so the quasi distribution in CG belongs to the universality class in LaMET;
- No self-energy of the Wilson link, improving the SNR significantly;
- Simplified renormalization $\bar{\psi}_0(z, b_{\perp}) \Gamma \psi_0(0) = Z_{\psi}(a) [\bar{\psi}(z, b_{\perp}) \Gamma \psi(0)]$;
- Larger off-axis momenta (3D rotational symmetry).



Gauge Invariant (GI)

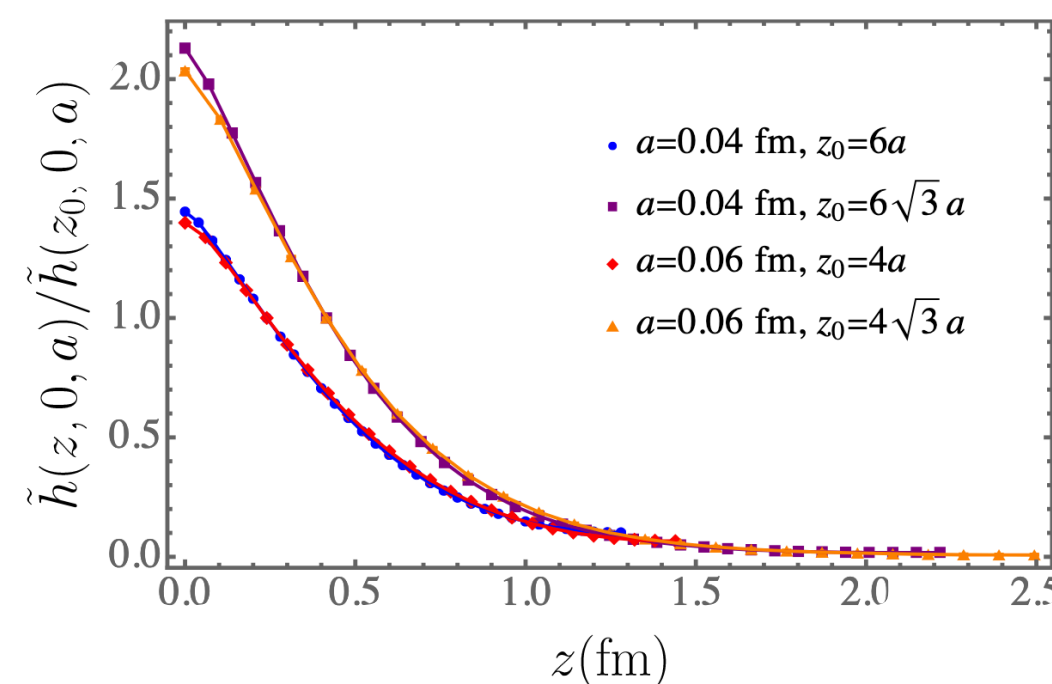
Coulomb Gauge (CG)

Collins-Soper kernel in CG v.s. GI

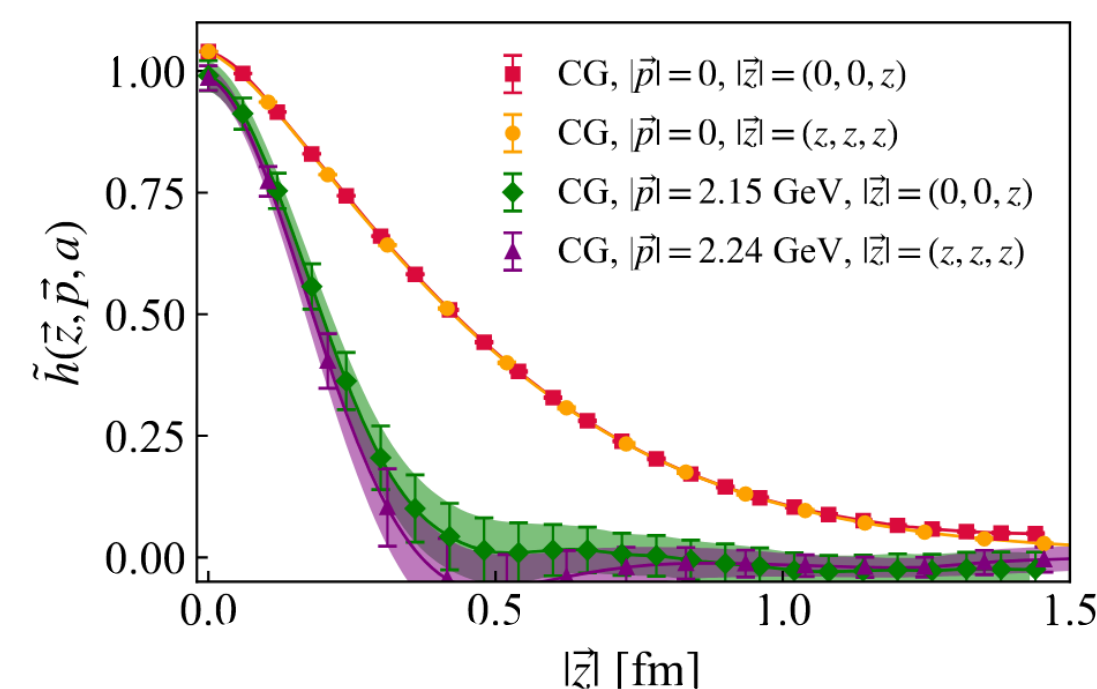


D. Bollweg, et al., Phys.Lett.B 852 (2024)

No linear divergence



3D rotational symmetry



X. Gao, W. Y. Liu and Y. Zhao, PRD 109 (2024)

Lattice Setup

- 2+1 flavor HISQ ensemble by HotQCD with volume $L_s \times L_t = 48^3 \times 64$;

- Lattice spacing is $a = 0.06$ fm;

- Pion mass of sea quark: $m_\pi^{\text{sea}} = 160$ MeV;

- Pion mass of valence quark for quasi-TMD: $m_\pi^{\text{val}} = 300$ MeV;

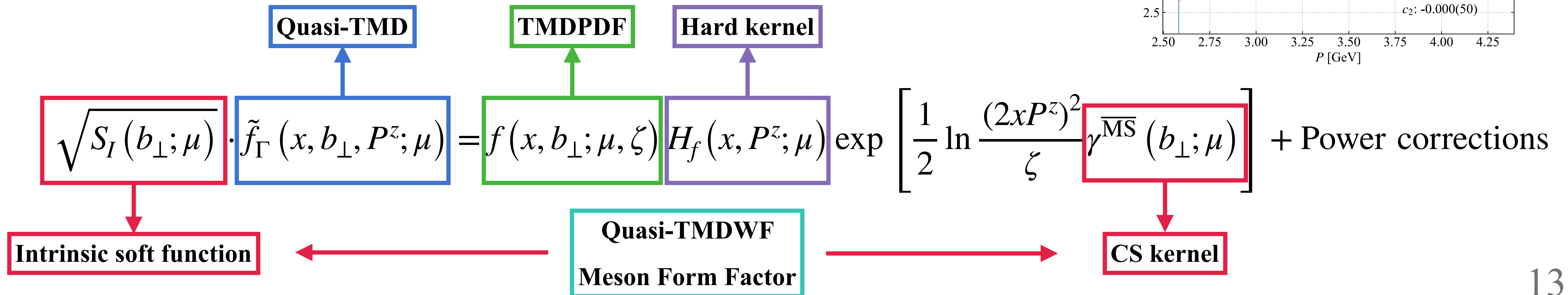
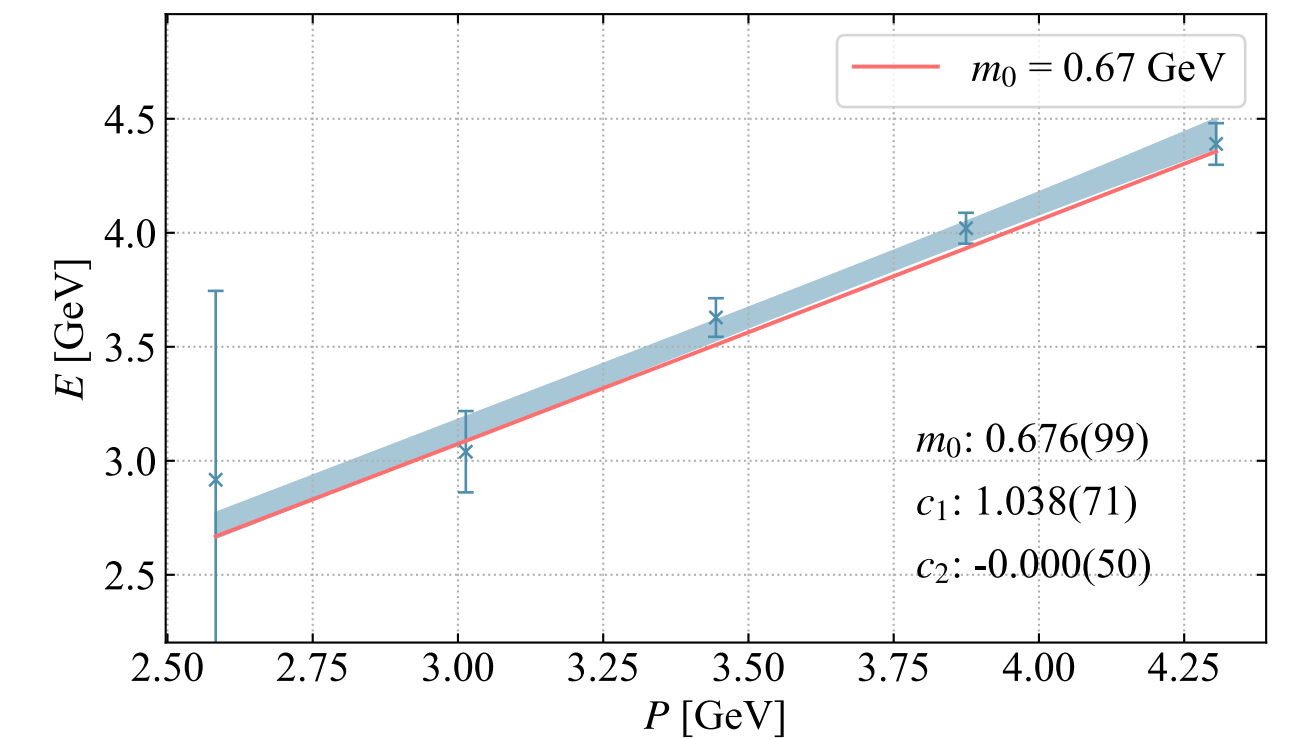
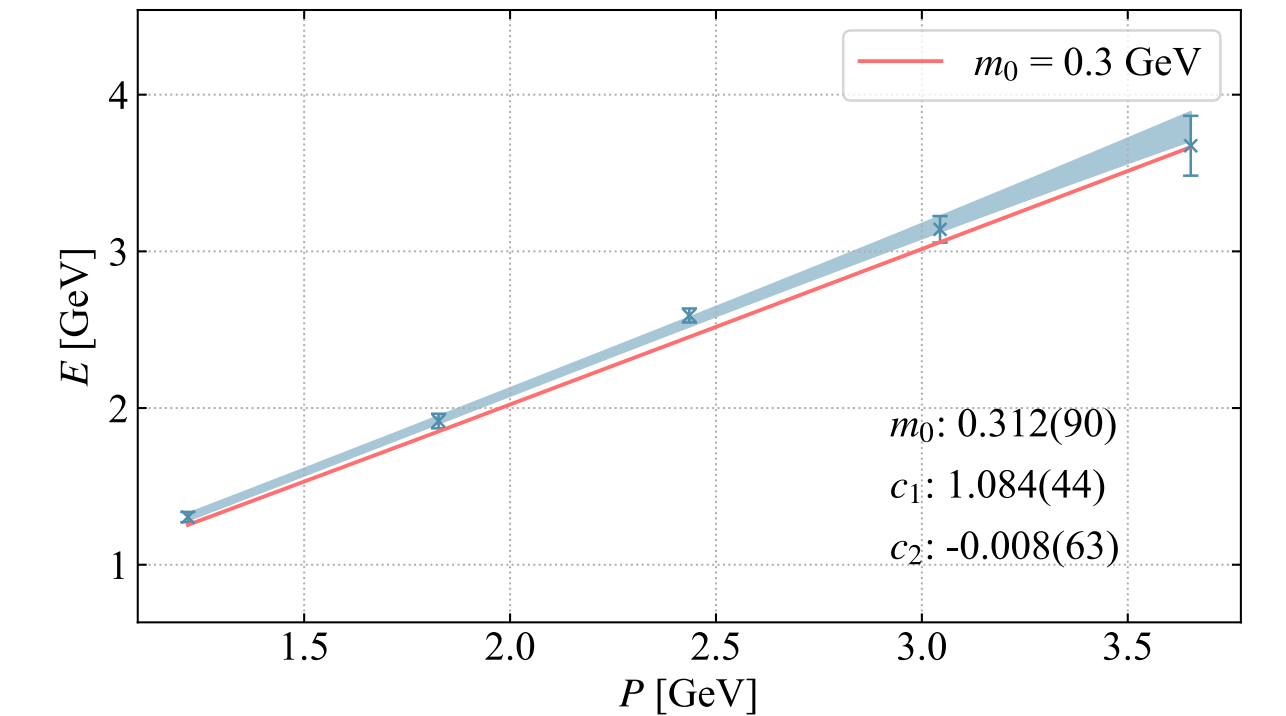
- Off-axis ($\vec{n} = (1,1,0)$) hadron momenta for quasi-TMD: 1.83 GeV, 2.43 GeV and 3.04 GeV;

- Pion mass of valence quark for qTMDWF and meson form factor: $m_\pi^{\text{val}} = 670$ MeV;

- On-axis hadron momenta for qTMDWF: 3.44 GeV, 3.87 GeV and 4.30 GeV;

- On-axis hadron momenta for meson form factor: 2.58 GeV.

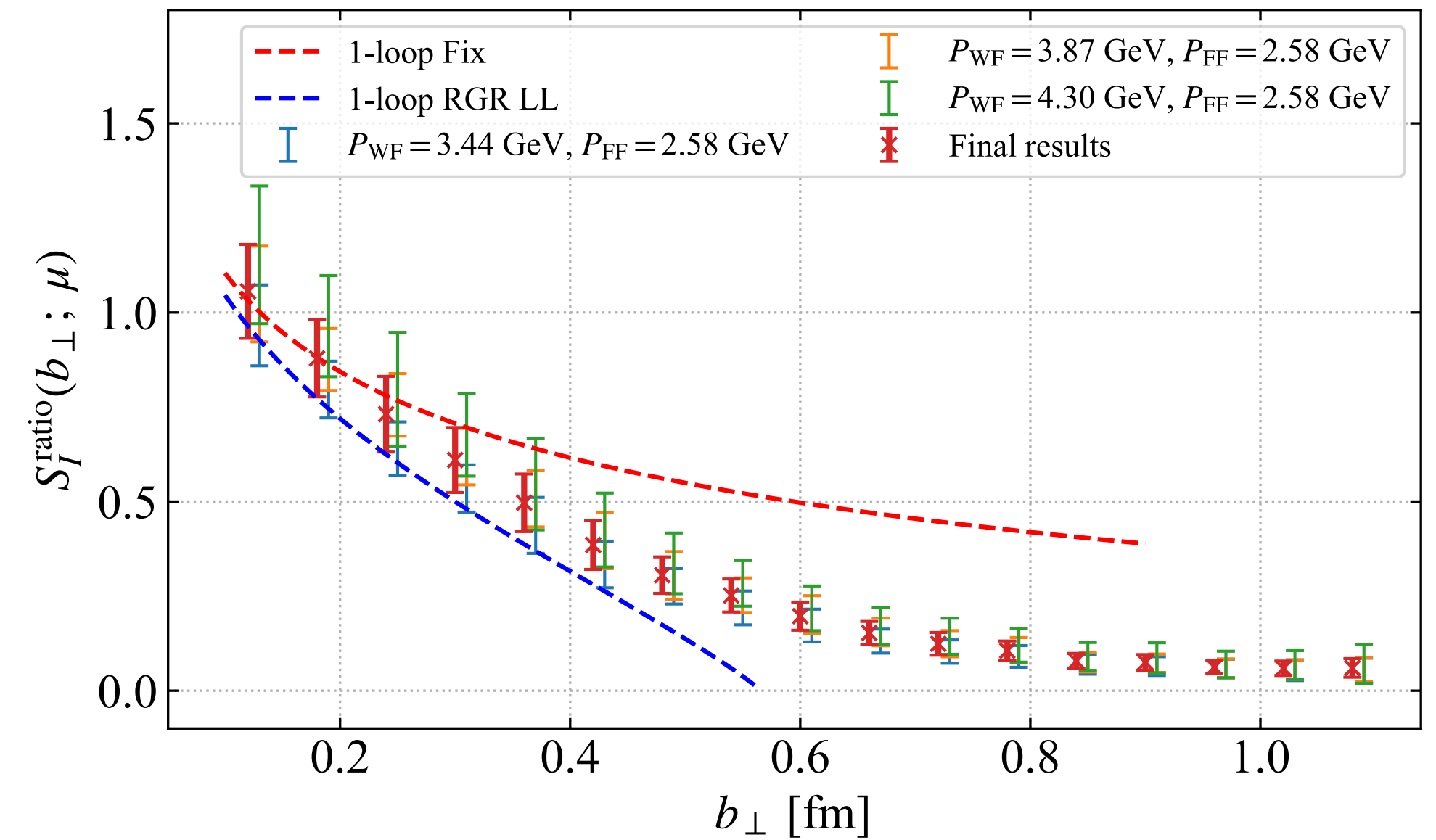
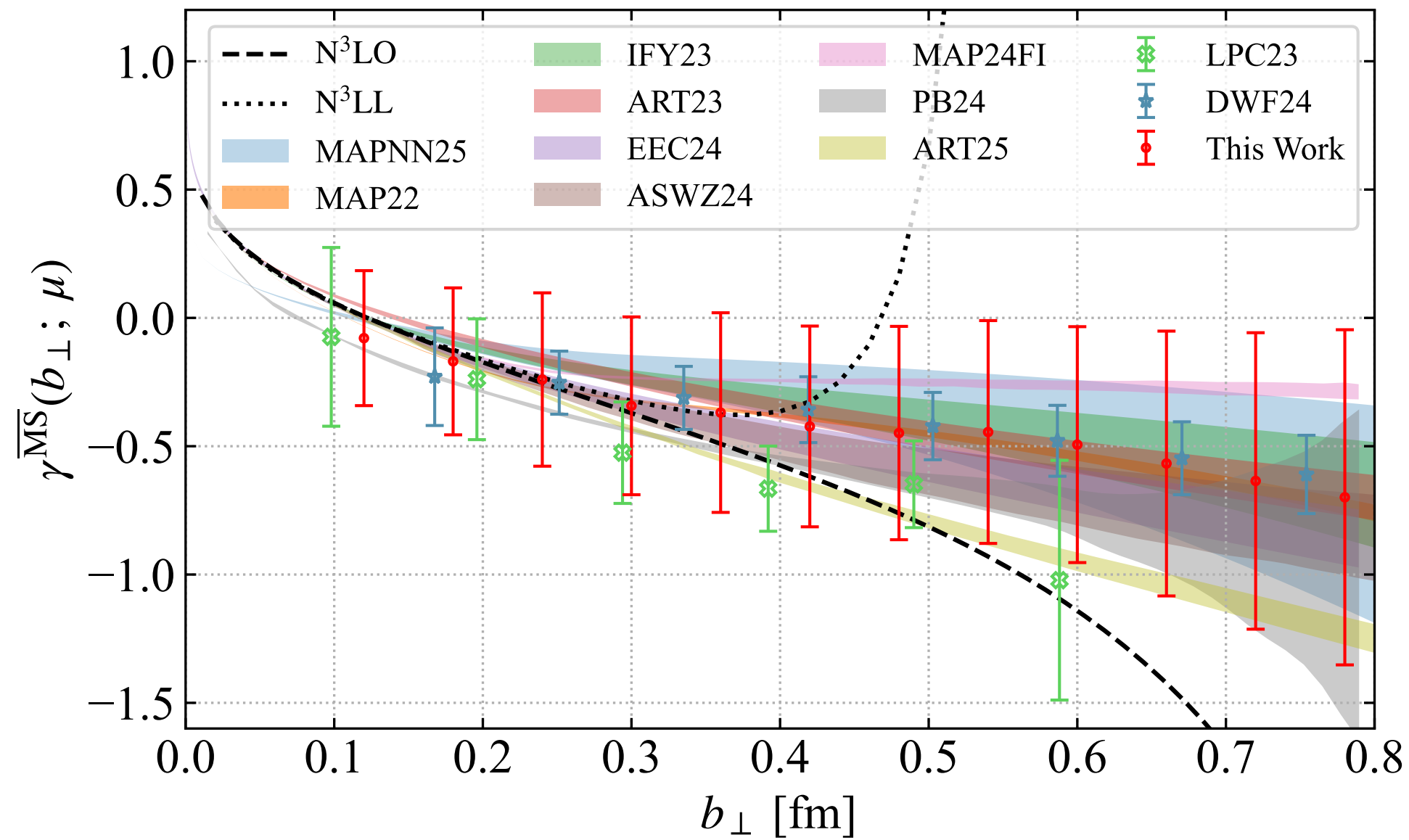
Dispersion relation: $E^2 = m_0^2 + c_1 P^2 + c_2 a^2 P^4$



CS Kernel & Intrinsic Soft Function

$$\gamma^{\overline{\text{MS}}}(b_{\perp}, P_1, P_2; \mu) = \frac{1}{\ln(P_2/P_1)} \ln \frac{H_{\phi}(x, \bar{x}, P_1; \mu) \tilde{\phi}_{\gamma^z \gamma^5}(x, b_{\perp}, P_2; \mu)}{H_{\phi}(x, \bar{x}, P_2; \mu) \tilde{\phi}_{\gamma^z \gamma^5}(x, b_{\perp}, P_1; \mu)}$$

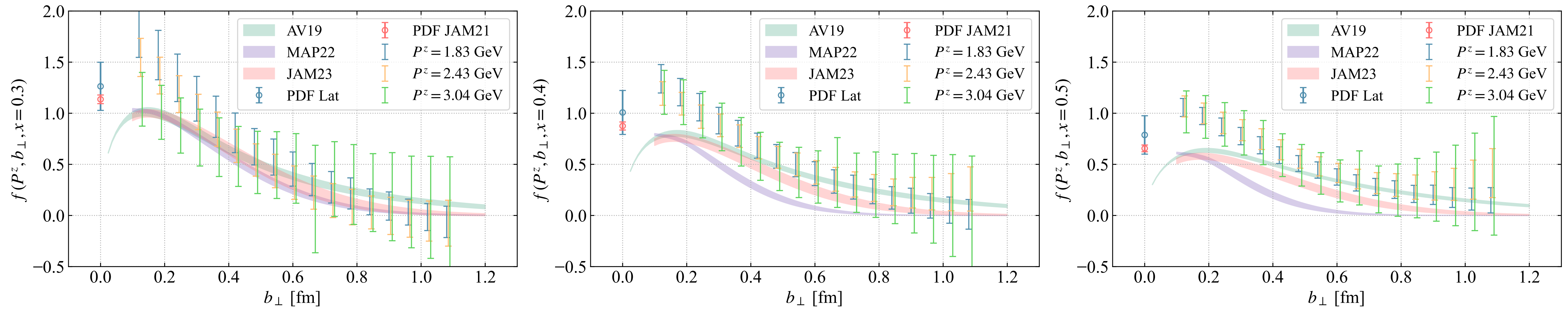
$$S_I(b_{\perp}; \mu) = \frac{F(b_{\perp}, P^z)}{\int dx_1 dx_2 H_F(x_1, x_2, P^z; \mu) \tilde{\Phi}^{\dagger}(x_1) \tilde{\Phi}(x_2)} \text{ with } \tilde{\Phi}(x) \equiv \frac{\tilde{\phi}_{\Gamma}(x, b_{\perp}, P^z; \mu)}{H_{\phi}(x, \bar{x}, P^z; \mu)}$$



Systematic uncertainty is estimated by the spread of the mean values between these three momentum pairs.

- There is a notable tension among recent results in phenomenology (MAP24FI & ART25);
- The large uncertainty is mainly caused by the **small Lorentz boost factor** at such a heavy pion mass ($m_{\pi}^{\text{val}} = 670 \text{ MeV}$);
- Our lattice results are consistent with the perturbation theory in the small $b_{\perp}$ regime;
- Thanks to the absence of linear divergence, our final results of the intrinsic soft function can go beyond $b_{\perp} \sim 1 \text{ fm}$.

Pion TMDPDF in the $b_\perp$ Space



A. Vladimirov, JHEP 10 (2019); M. Cerutti, et al. (MAP), Phys.Rev.D 107 (2023); P. C. Barry, et al. (JAM) Phys.Rev.D 108 (2023)

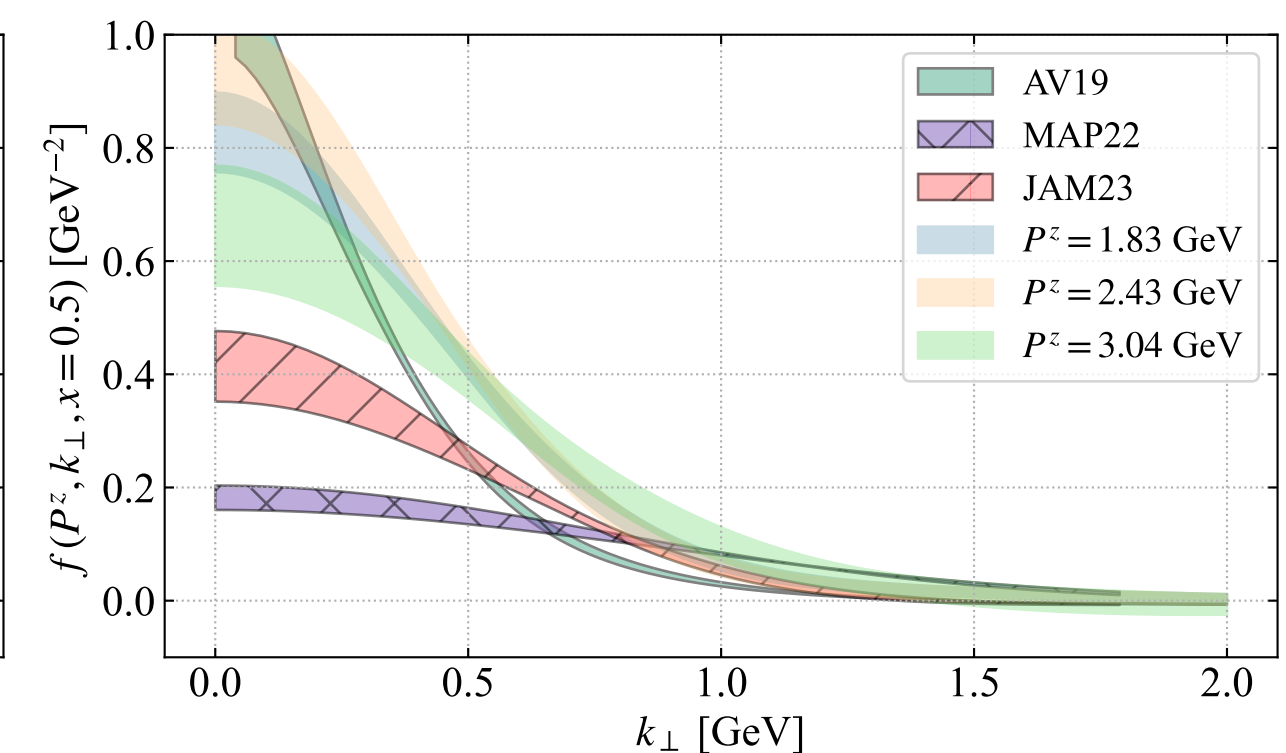
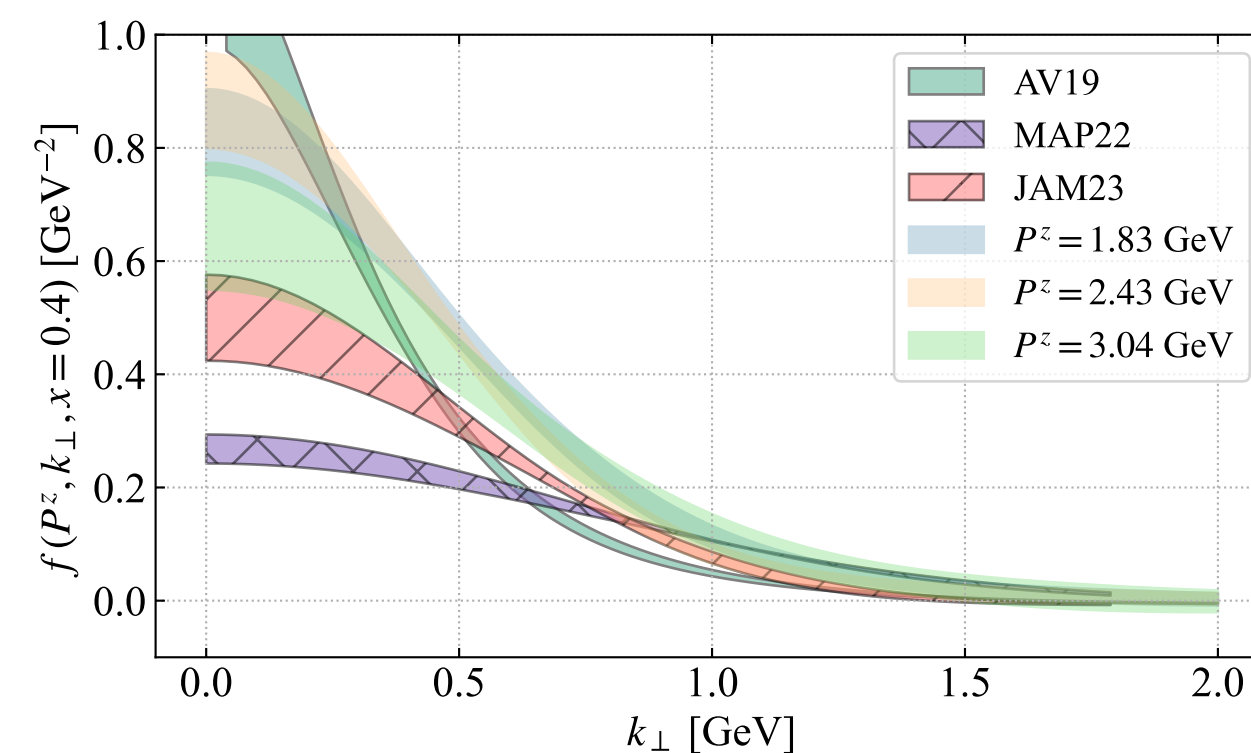
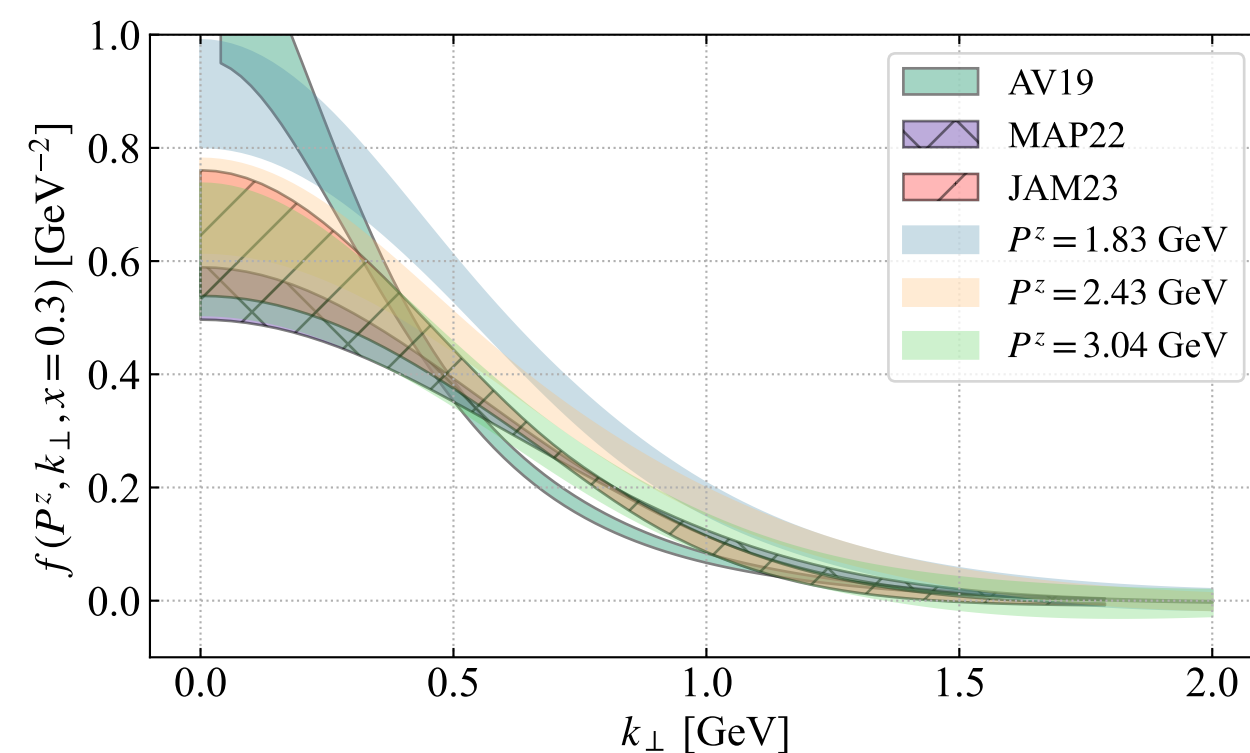
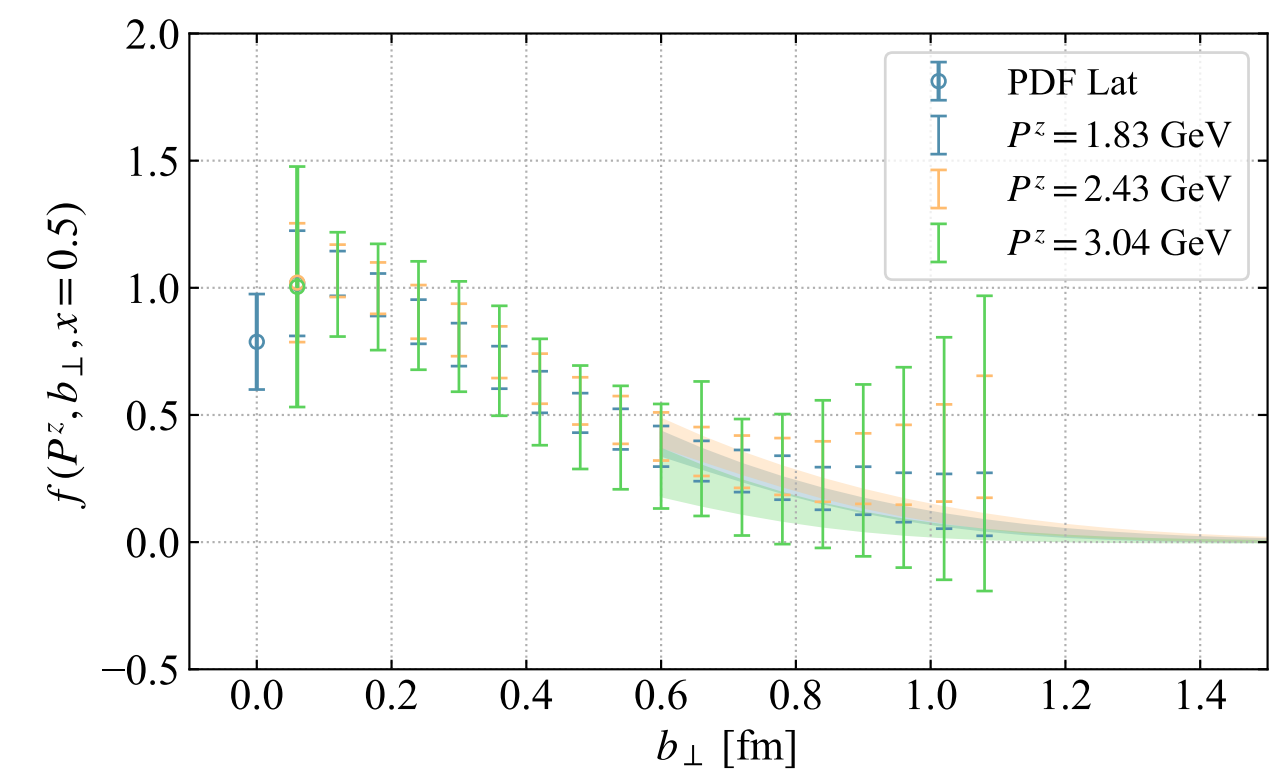
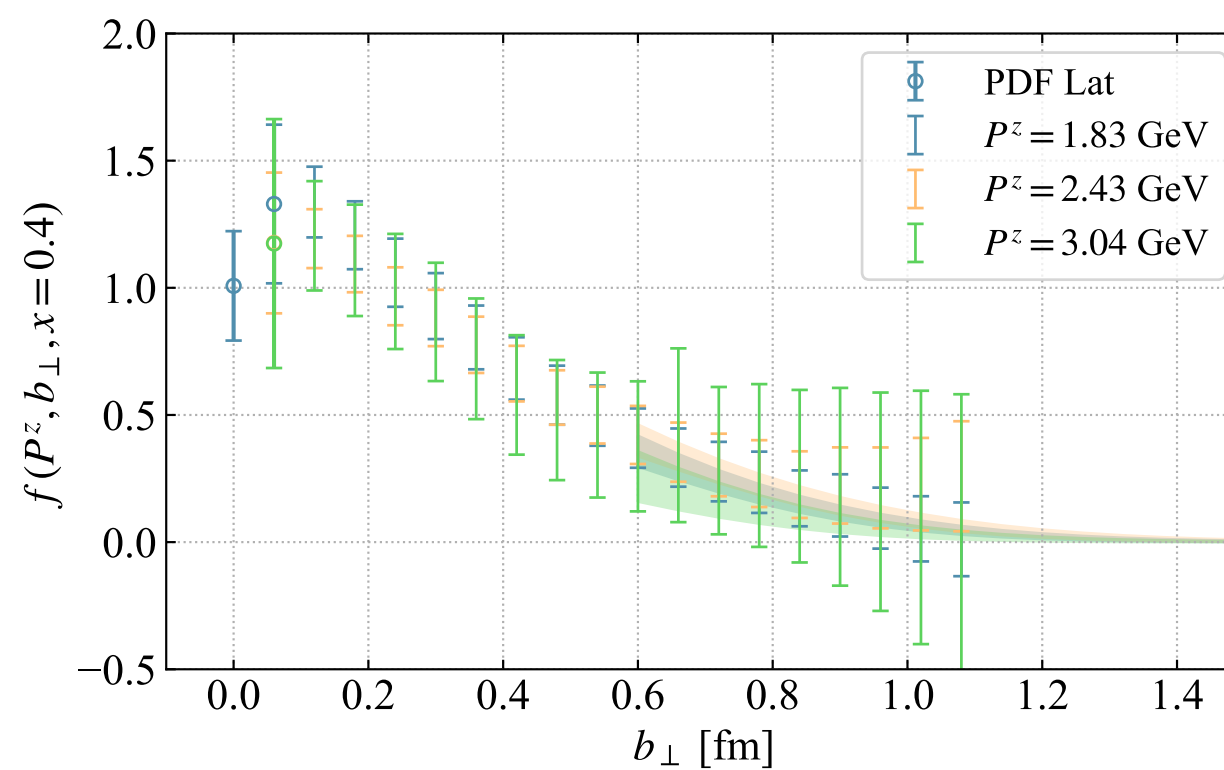
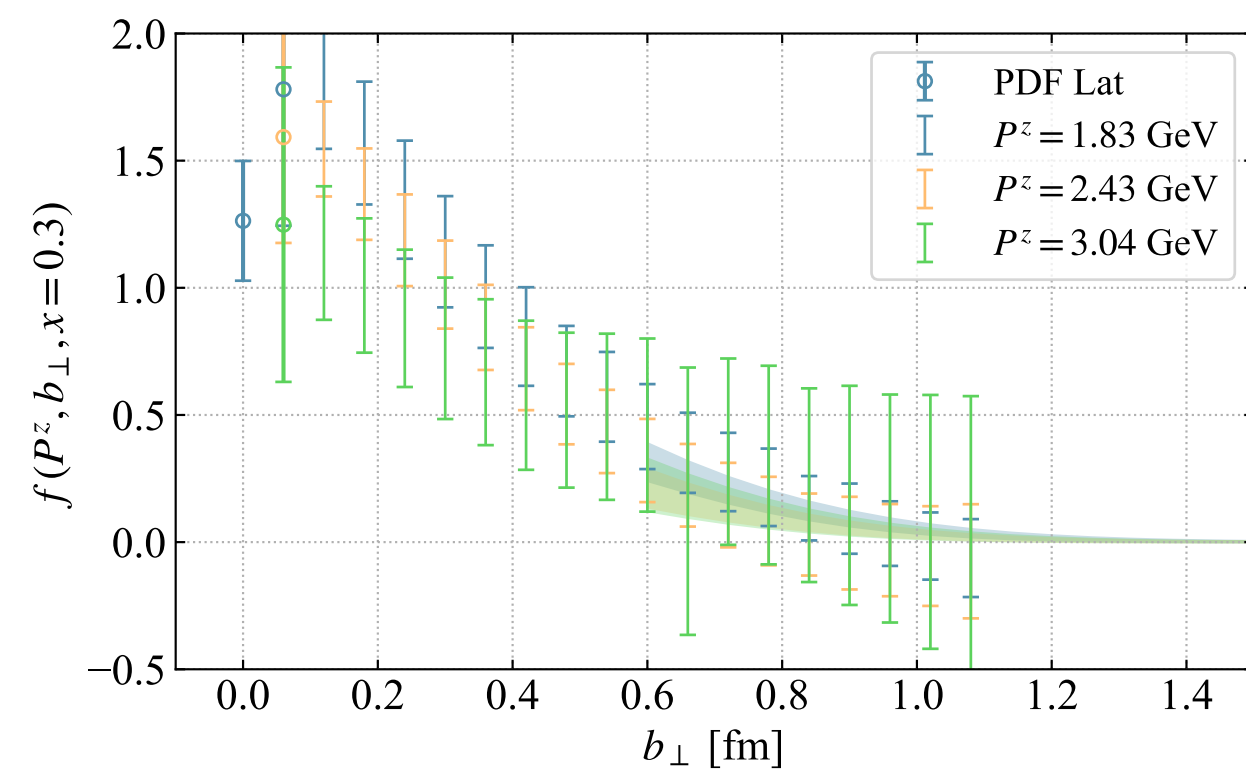
- Thanks to the absence of linear divergence, we can calculate pion TMDPDF up to $b_\perp > 1$ fm;
- When x gets larger, the amplitude of TMDPDF is decreasing, while the the transverse correlation length stays roughly the same;
- When x gets closer to $x = 0.5$, we can find that the variance across different momenta becomes smaller, indicating the suppression of power correction;
- Global analysis has a better control at relatively small x , where we saw consistency;
- Lattice provides predictions at relatively large x , where experimental data gives less constraints.

Pion TMDPDF in the $k_{\perp}$ Space

- We can give the $k_{\perp}$ -dependence thanks to the good SNR in CG;

- Extrapolate the large $b_{\perp}$ using a simple Gaussian form: $f(b_{\perp}) = Ae^{-mb_{\perp}^2}$;

- Fourier transform to the $k_{\perp}$ space: $\tilde{f}(k_{\perp}) = \int \frac{d^2\vec{b}_{\perp}}{(2\pi)^2} e^{i\vec{b}_{\perp} \cdot \vec{k}_{\perp}} f(b_{\perp}) = \int \frac{d|b_{\perp}|}{2\pi} |b_{\perp}| \cdot J_0(|b_{\perp}| \cdot |k_{\perp}|) \cdot f(b_{\perp})$.



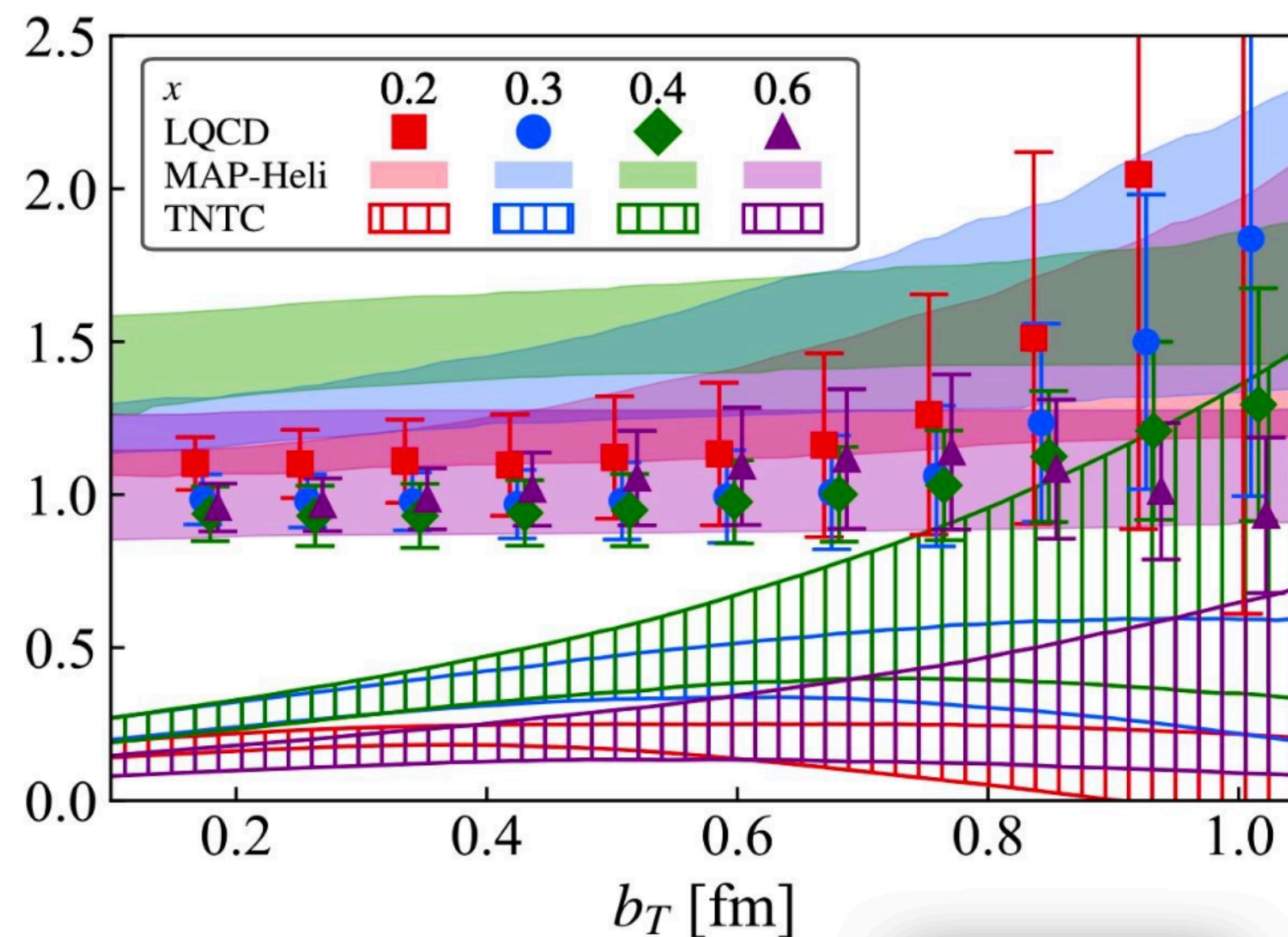
Summary

- This is the **first lattice calculation** of the pion unpolarized TMDPDF within LaMET framework;
- The **novel CG** method is employed to remove the linear divergence, so that to have a good SNR up to $b_{\perp} > 1$ fm;
- The soft function is extracted at **NLL factorization** using RG resummation, the results show consistency with perturbation theory;
- The TMDs, including CS kernel, intrinsic function, TMDWF and TMDPDF are calculated using the same lattice ensemble, and the results show consistency with existing studies, including phenomenology and lattice calculations;
- The outcome of this study highlights the efficacy of the CG quasi-TMD approach in probing the transverse momentum structure of hadrons;
- In future work, we will apply the CG approach to the nucleon, and lattice systematics such as discretization effects and the unphysical pion mass will be investigated in detail.

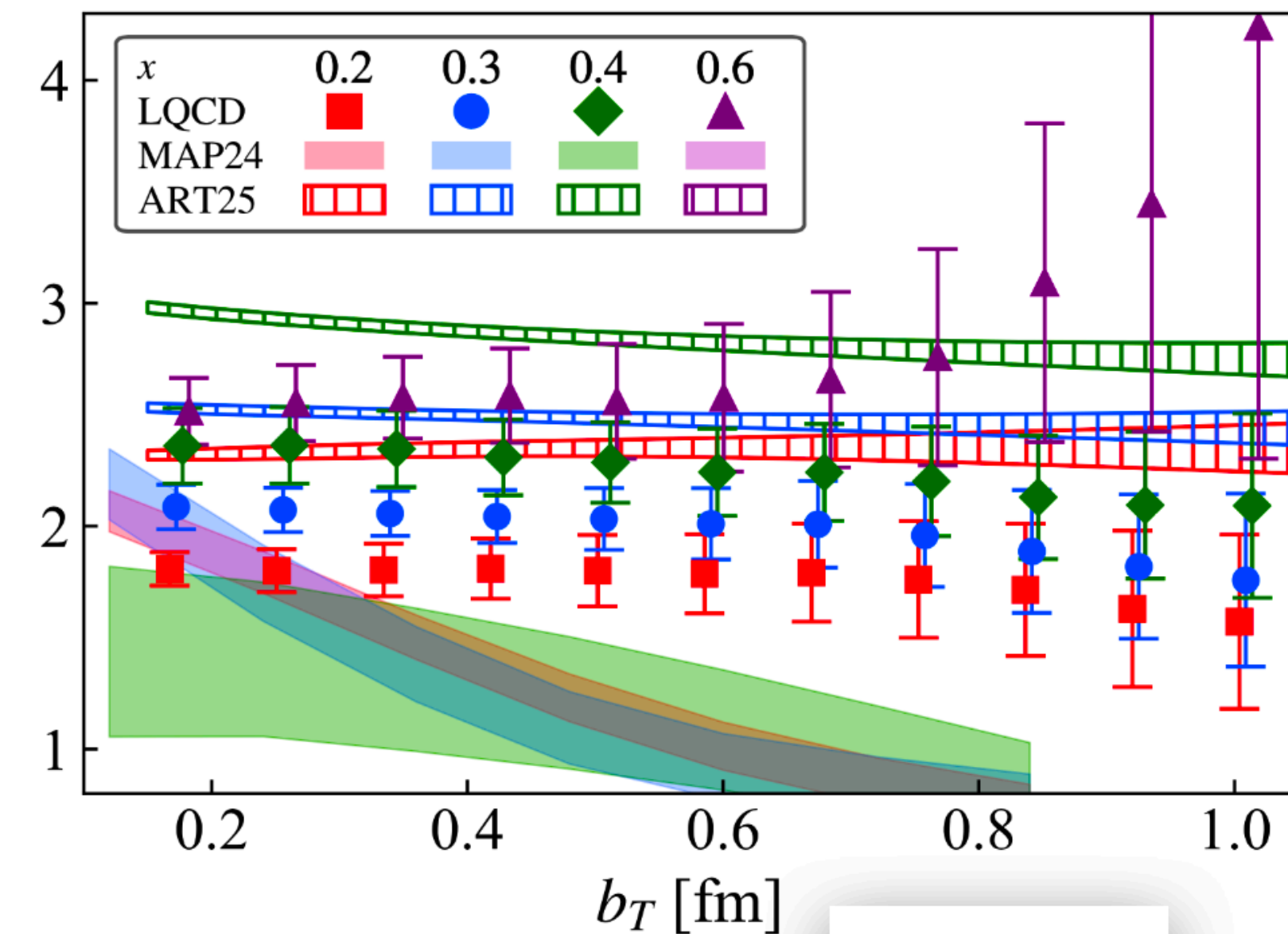
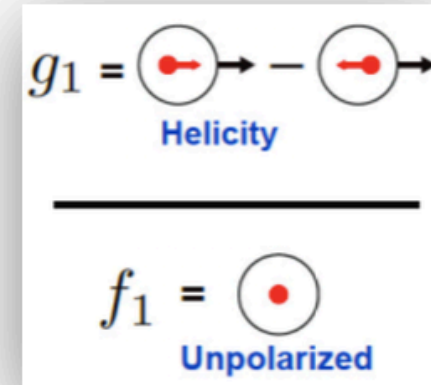
CG Method for Nucleon

D. Bollweg, X. Gao, S. Mukherjee, Y. Zhao, PRL 135 (2025)

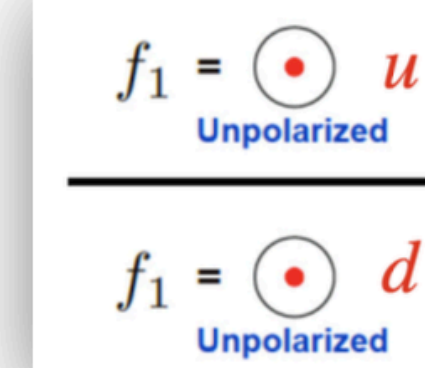
○ Ratios between nucleon TMDPDFs



$$\frac{g_{1L}^{\Delta u^+ - \Delta d^+}(x, b_T)}{f_1^{u_v - d_v}(x, b_T)} \frac{1}{g_A}$$



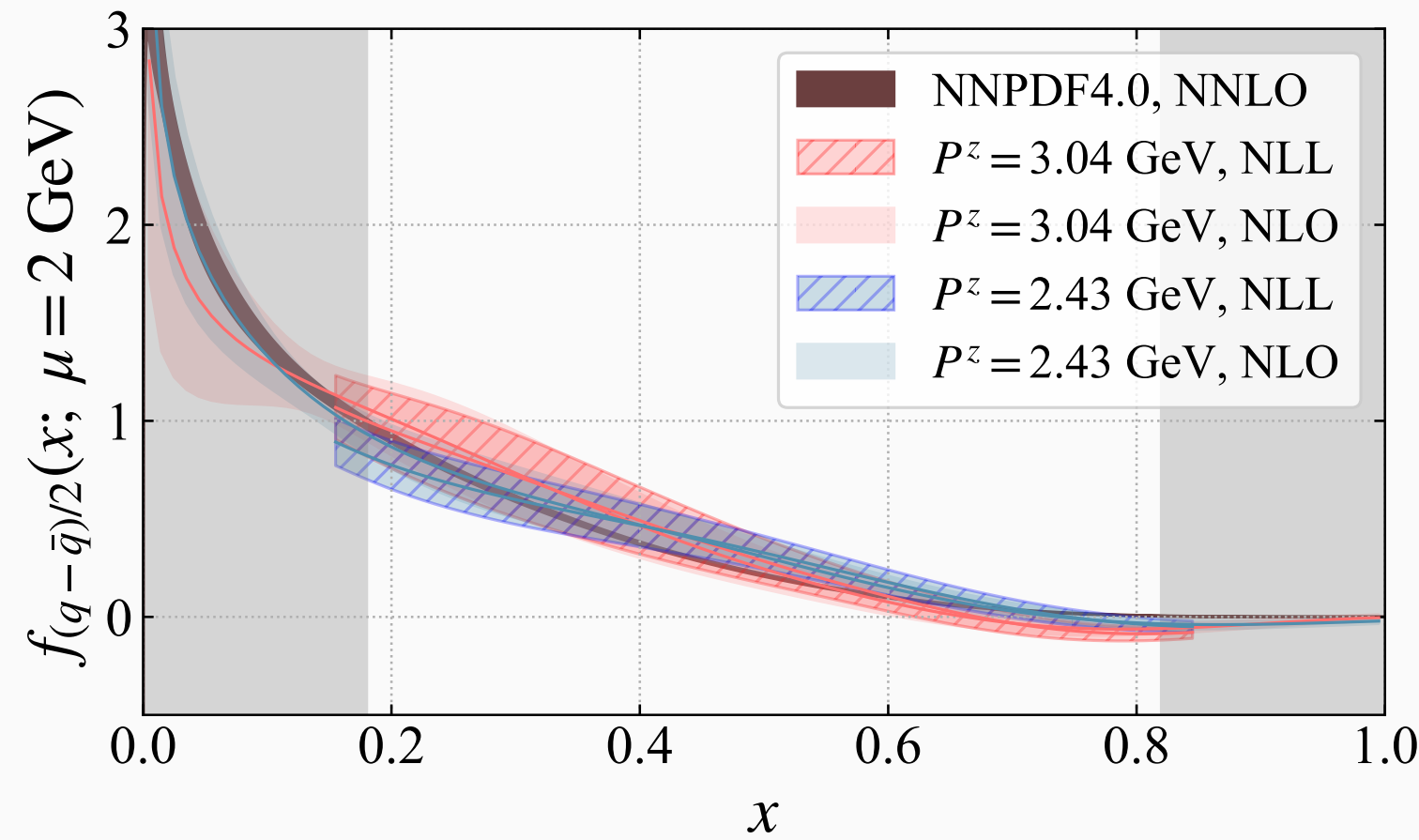
$$\frac{f_1^{u_v}(x, b_T)}{f_1^{d_v}(x, b_T)}$$



Nucleon PDFs from Coulomb Gauge Method

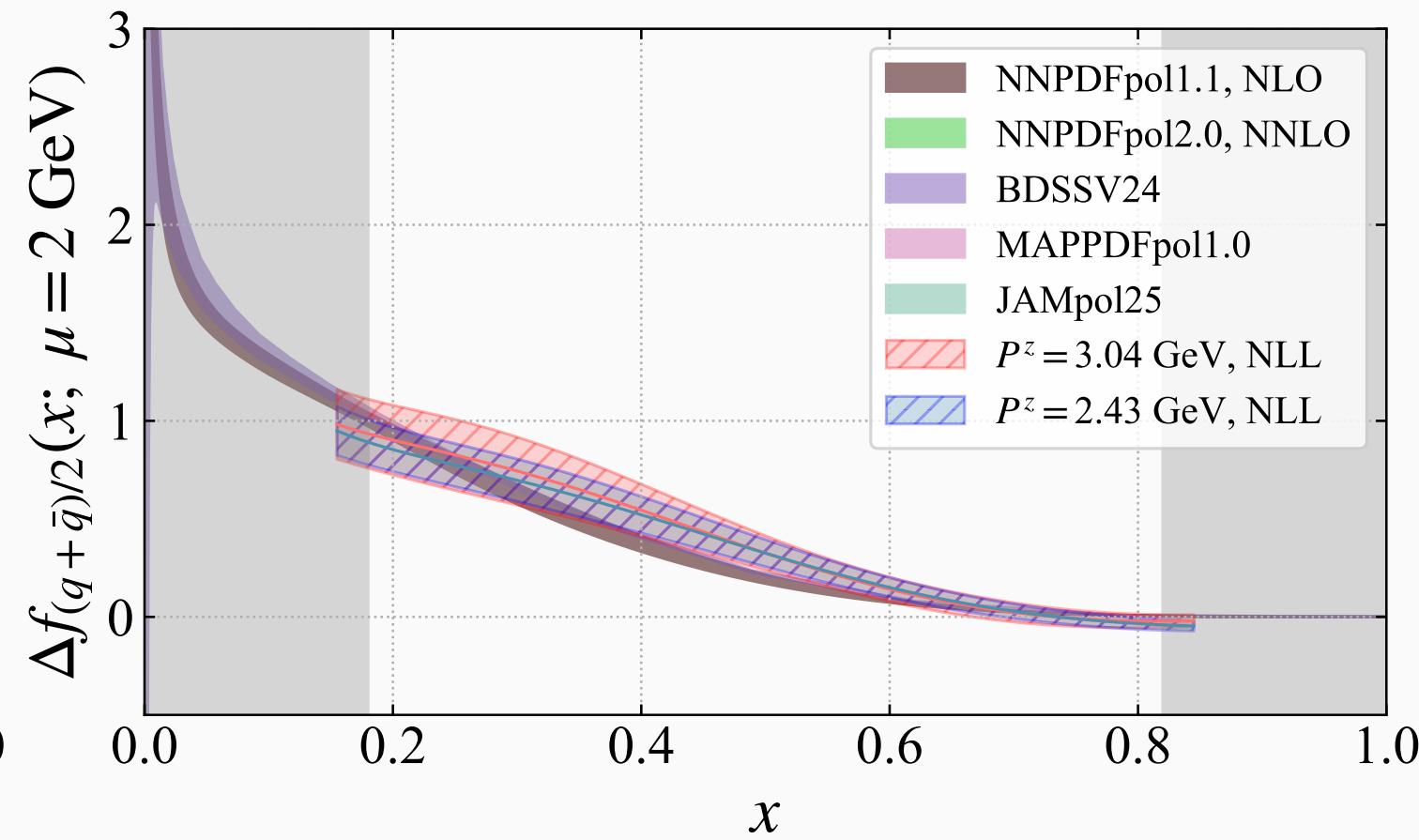
X. Gao, JH, et al., 2602.11283

Unpolarized PDF $(q - \bar{q})/2$



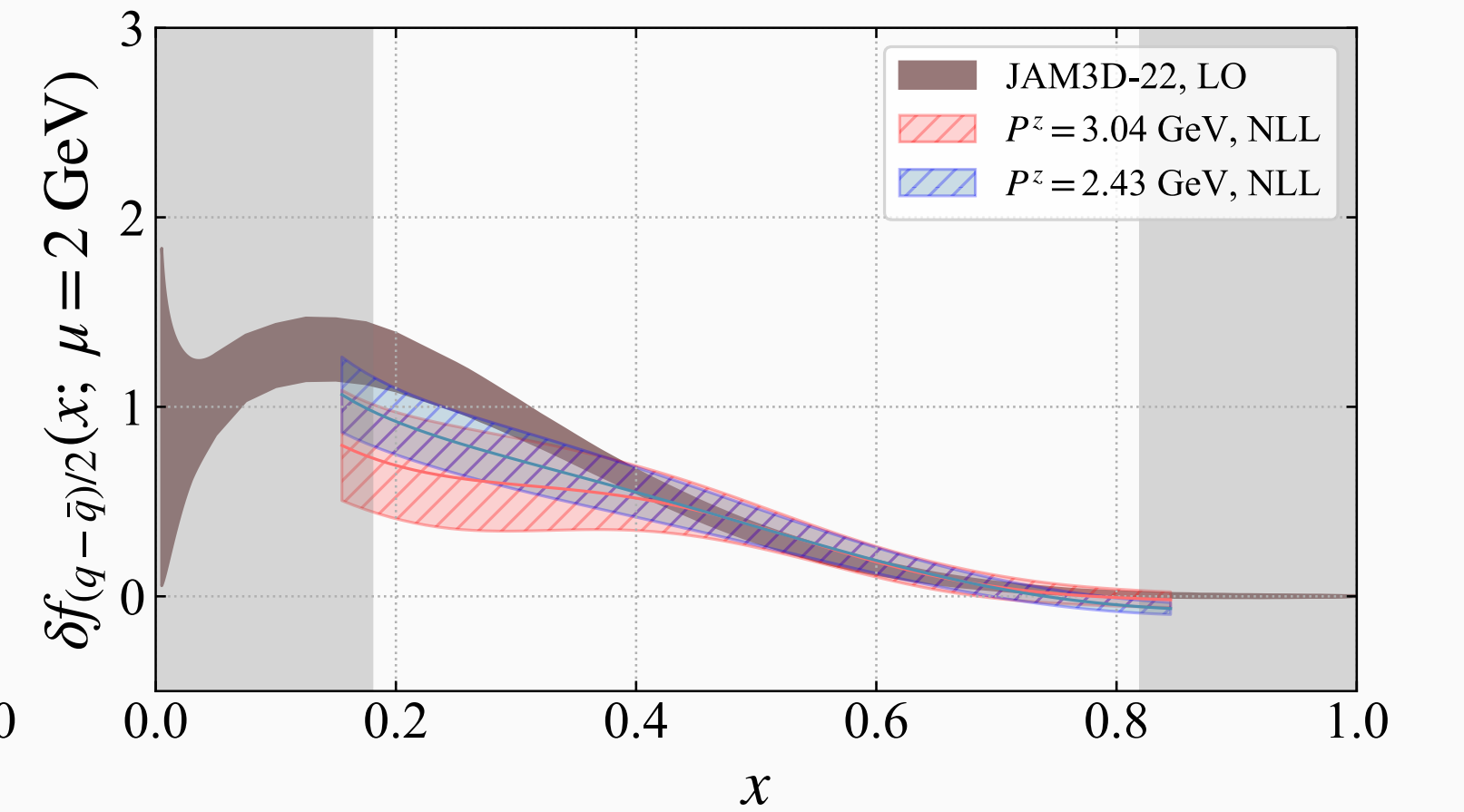
NNPDF4.0: R. D. Ball, et al. [NNPDF], EPJC 82 (2022)

Helicity PDF $(q + \bar{q})/2$



NNPDFpol1.1: E. R. Nocera et al. [NNPDF], NPB 887 (2014)
 NNPDFpol2.0: J. Cruz-Martinez et al. [NNPDF], JHEP 07 (2025)
 BDSSV24: I. Borsa et al., PRL 133 (2024)
 MAPPDFpol1.0: V. Bertone et al. [MAP], PLB 865 (2025)
 JAMpol25: C. Cocuzza et al. [JAM], 2506.13616

Transversity PDF $(q - \bar{q})/2$



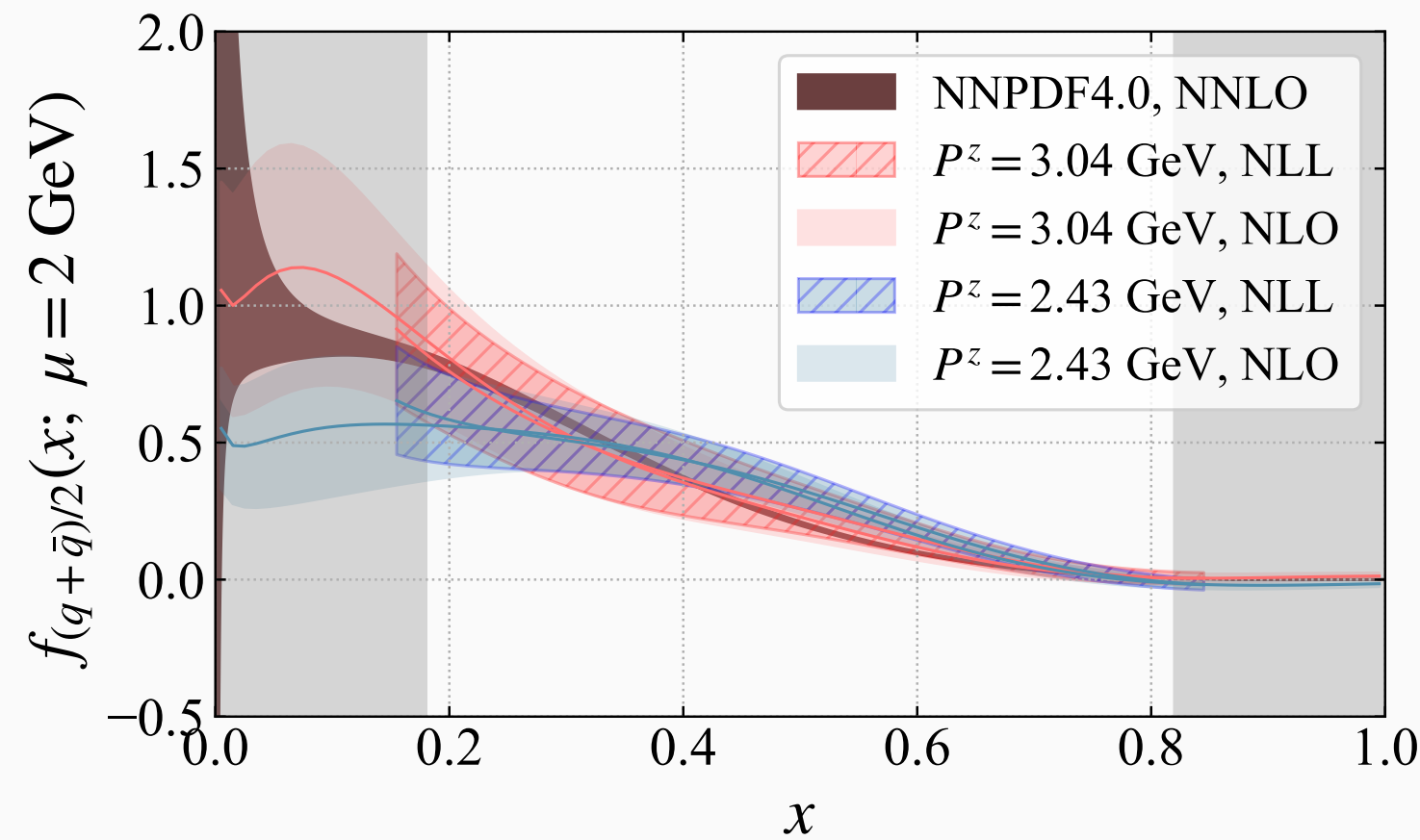
JAM3D-22: L. Gamberg, et al. [JAM], PRD 106 (2022)

- This work presents the **first lattice calculation** of the nucleon PDFs for all three polarizations using the **CG method**;
- The results of **real part contribution** in all three PDFs show encouraging agreement with existing results at **moderate x region**;
- The **NLO** and **NLL with RGR** results show an aligned behavior at **moderate x**, where LaMET can make reliable prediction;

Nucleon PDFs from Coulomb Gauge Method

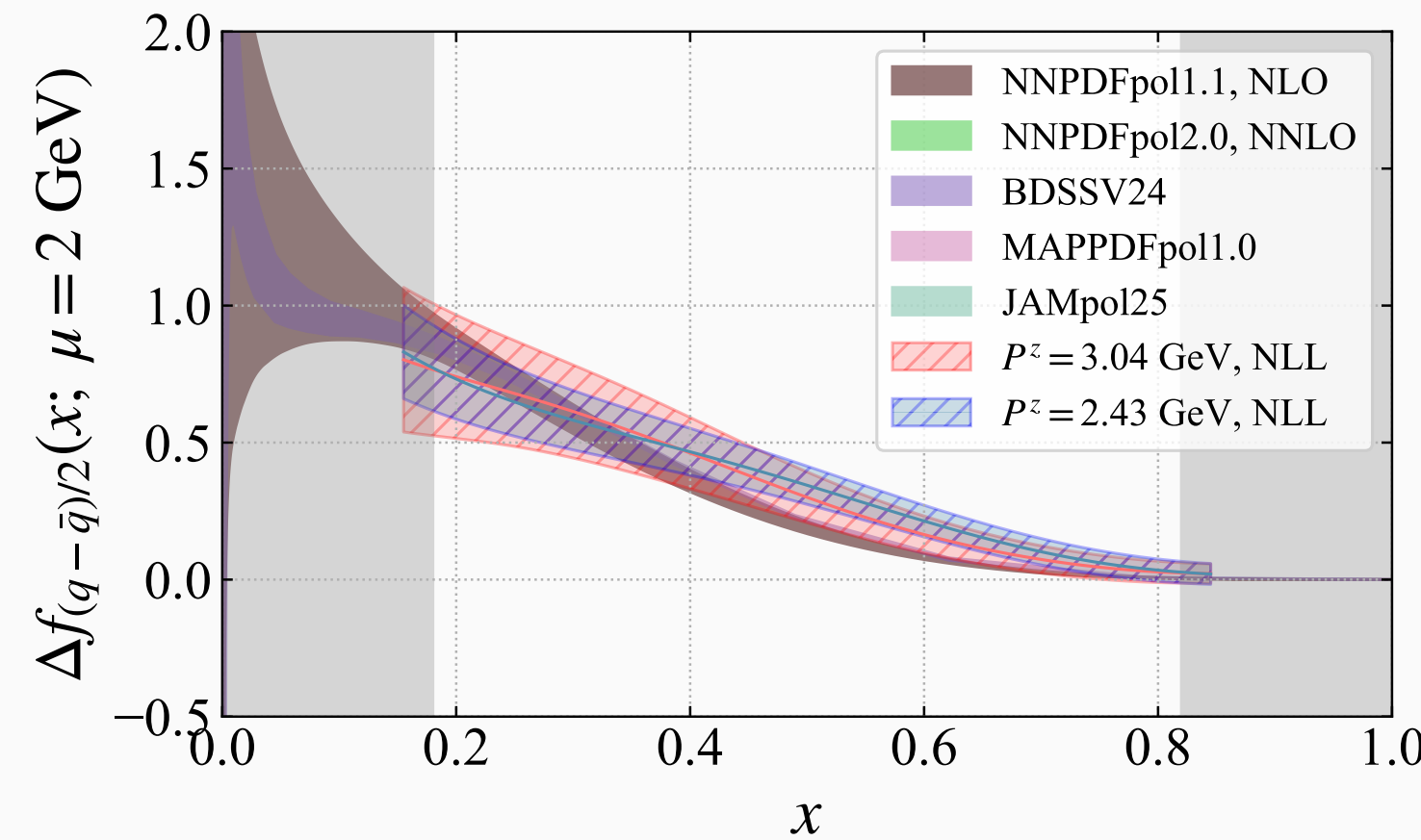
X. Gao, JH, et al., 2602.11283

Unpolarized PDF $(q + \bar{q})/2$



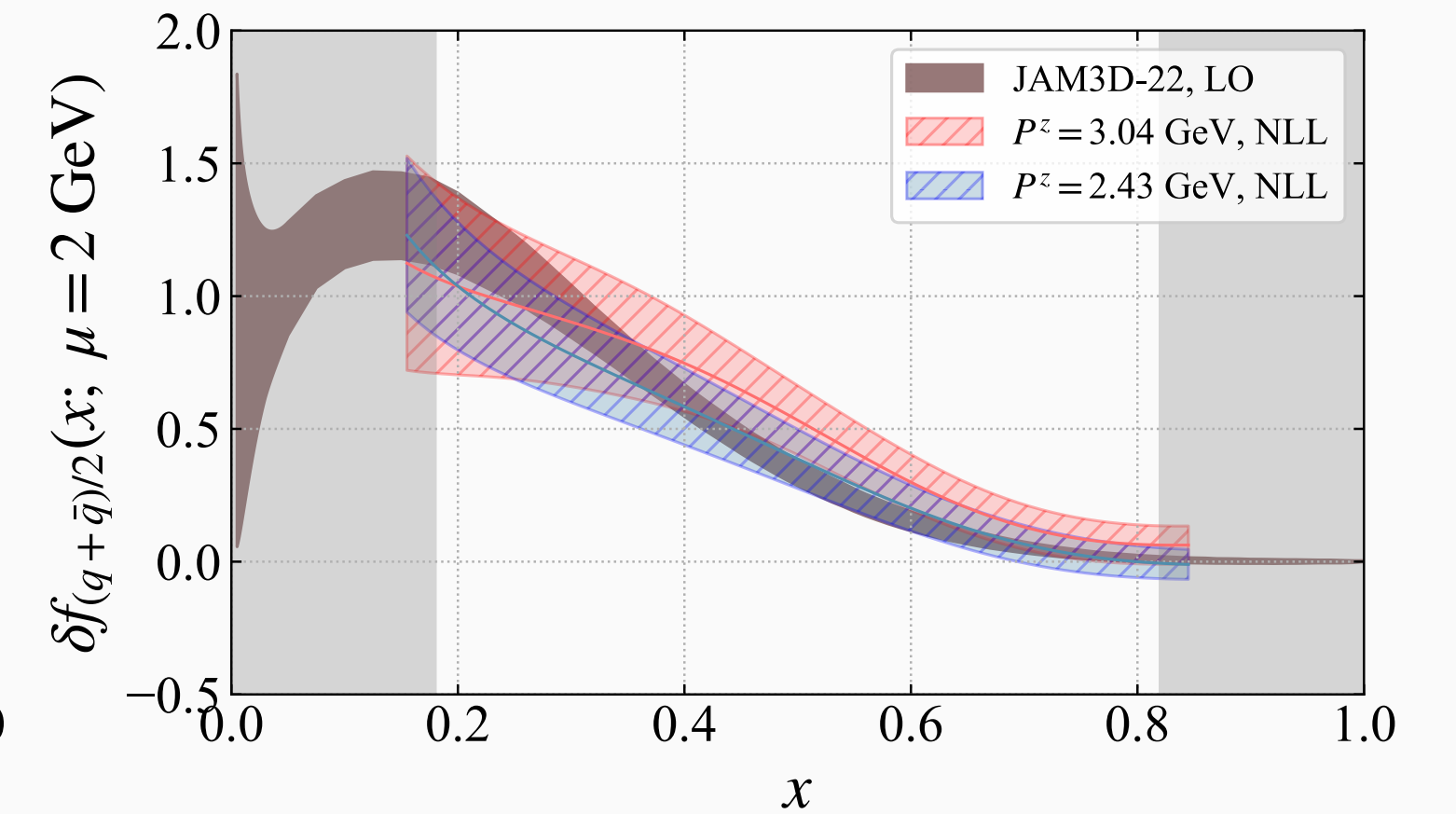
NNPDF4.0: R. D. Ball, et al. [NNPDF], EPJC 82 (2022)

Helicity PDF $(q - \bar{q})/2$



NNPDFpol1.1: E. R. Nocera et al. [NNPDF], NPB 887 (2014)
 NNPDFpol2.0: J. Cruz-Martinez et al. [NNPDF], JHEP 07 (2025)
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 JAMpol25: C. Cocuzza et al. [JAM], 2506.13616

Transversity PDF $(q + \bar{q})/2$



JAM3D-22: L. Gamberg, et al. [JAM], PRD 106 (2022)

- For the imaginary part, the **discrepancy between two different momenta** is larger than that for the real part, possibly caused by excited-state contamination;
- In GI cases, it is also found that imaginary part of matrix elements is more sensitive to excited-state contamination.

C. Alexandrou, et al., Phys.Rev.D 103 (2021)

Backup

Soft Function

$$\sqrt{S_I(b_\perp; \mu)} \tilde{f}_\Gamma(x, b_\perp, P^z; \mu) = f(x, b_\perp; \mu, \zeta) H_f(x, P^z; \mu) \exp \left[\frac{1}{2} \ln \frac{(2xP^z)^2}{\zeta} \gamma^{\overline{\text{MS}}}(b_\perp; \mu) \right] + \text{Power corrections}$$

After regularization, the rapidity evolution is controlled by Collins-Soper scale: $\zeta = 2(xP^+)^2 e^{-2y_n}$

- The soft gluon radiation will lead to the existence of **soft functions**;
- Due to the factorization of the soft mode, the soft function contains the well-

known rapidity divergence;

$$\underbrace{\int_{q_T}^Q \frac{dk}{k}}_{\text{full}} = \lim_{\tau \rightarrow 0} \left[\underbrace{\int_0^Q \frac{dk}{k} R_c(k, \tau)}_{\text{collinear}} + \underbrace{\int_{q_T}^\infty \frac{dk}{k} R_s(k, \tau)}_{\text{soft}} \right] = \ln \frac{Q}{q_T}$$

- The soft function can be separated into two parts:

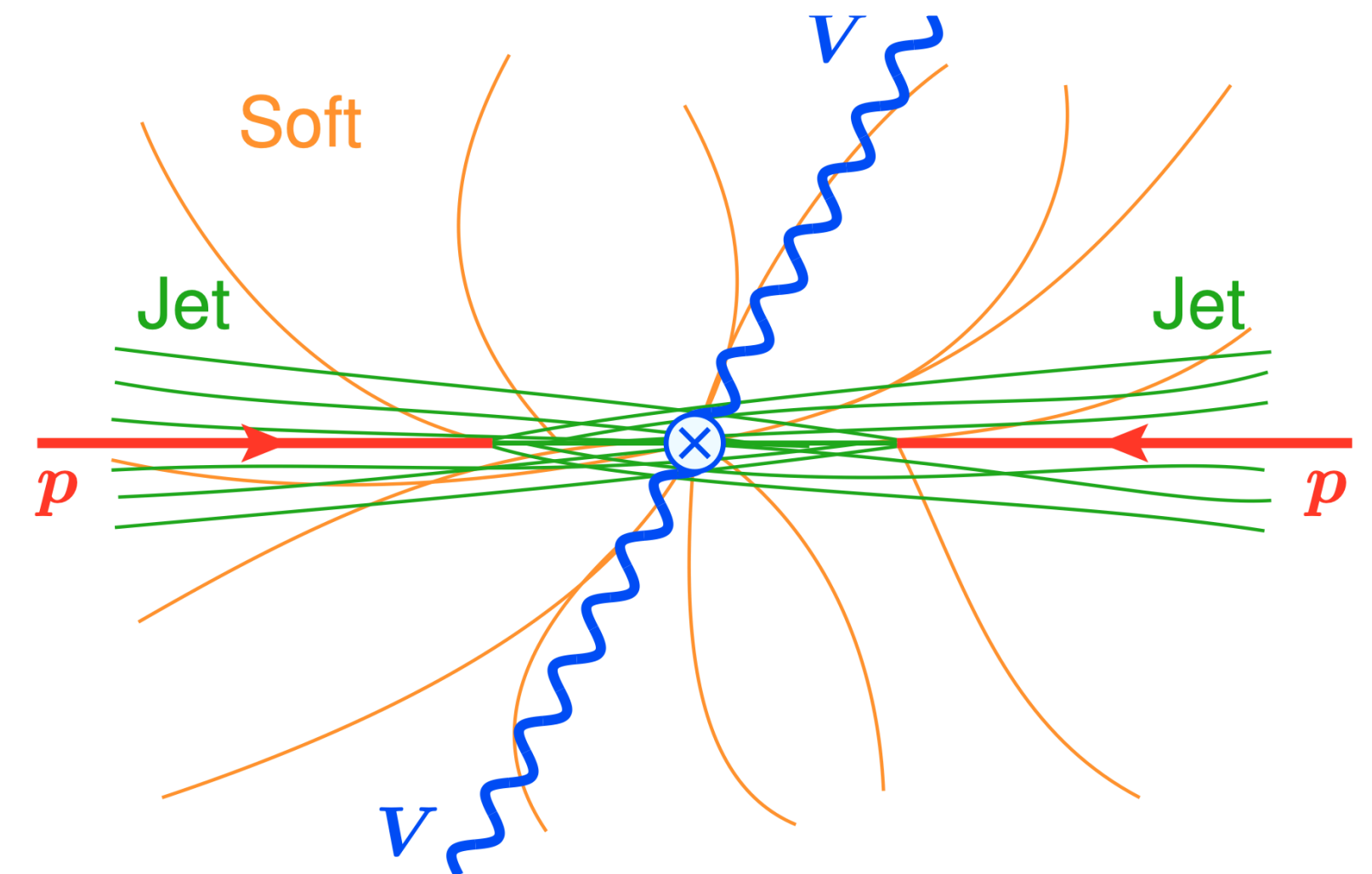
- Rapidity evolution kernel: CS kernel $\gamma^{\overline{\text{MS}}}(b_\perp; \mu)$ $\gamma^{\overline{\text{MS}}}(b_\perp; \mu) = 2 \frac{d}{d \ln \zeta} \ln f(x, b_\perp; \mu, \zeta)$

- Rapidity independent part: intrinsic soft function $S_I(b_\perp; \mu)$

- CS kernel can be extracted from the rapidity evolution of TMDs

$$\gamma^{\overline{\text{MS}}}(b_\perp, P_1, P_2; \mu) = \frac{1}{\ln(P_2/P_1)} \ln \frac{H_f(x, \bar{x}, P_1; \mu) \tilde{f}_{\gamma^t}(x, b_\perp, P_2; \mu)}{H_f(x, \bar{x}, P_2; \mu) \tilde{f}_{\gamma^t}(x, b_\perp, P_1; \mu)}$$

$H_f(x, P^z; \mu) = |C_{\text{TMD}}(xP^z; \mu)|^2$ is the TMD hard kernel for matching.



M. Ebert, PhD Thesis (2017)

Soft Function

- The intrinsic soft function cannot be directly calculated on lattice because of two light-like Wilson lines in different directions

X. Ji, Y. Liu, Y. Liu, Nucl.Phys.B 955 (2020)

- Fortunately, it can be extracted from the meson form factor

$$F(b_{\perp}, P_1, P_2, \Gamma, \Gamma') \equiv -4N_c \frac{\langle P_2 | \bar{q}(b_{\perp}) \Gamma q(b_{\perp}) \bar{q}(0) \Gamma' q(0) | P_1 \rangle}{f_{\pi}^2(P_1 \cdot P_2)}$$

- The form factor satisfies the factorization formula

$$F(b_{\perp}, P^z) = \int dx_1 dx_2 H_F(x_1, x_2, P^z; \mu) \phi^{\dagger}(x_1, b_{\perp}, y_n; \mu, \zeta_1, \bar{\zeta}_1) \phi(x_2, b_{\perp}, -y_n; \mu, \zeta_2, \bar{\zeta}_2)$$

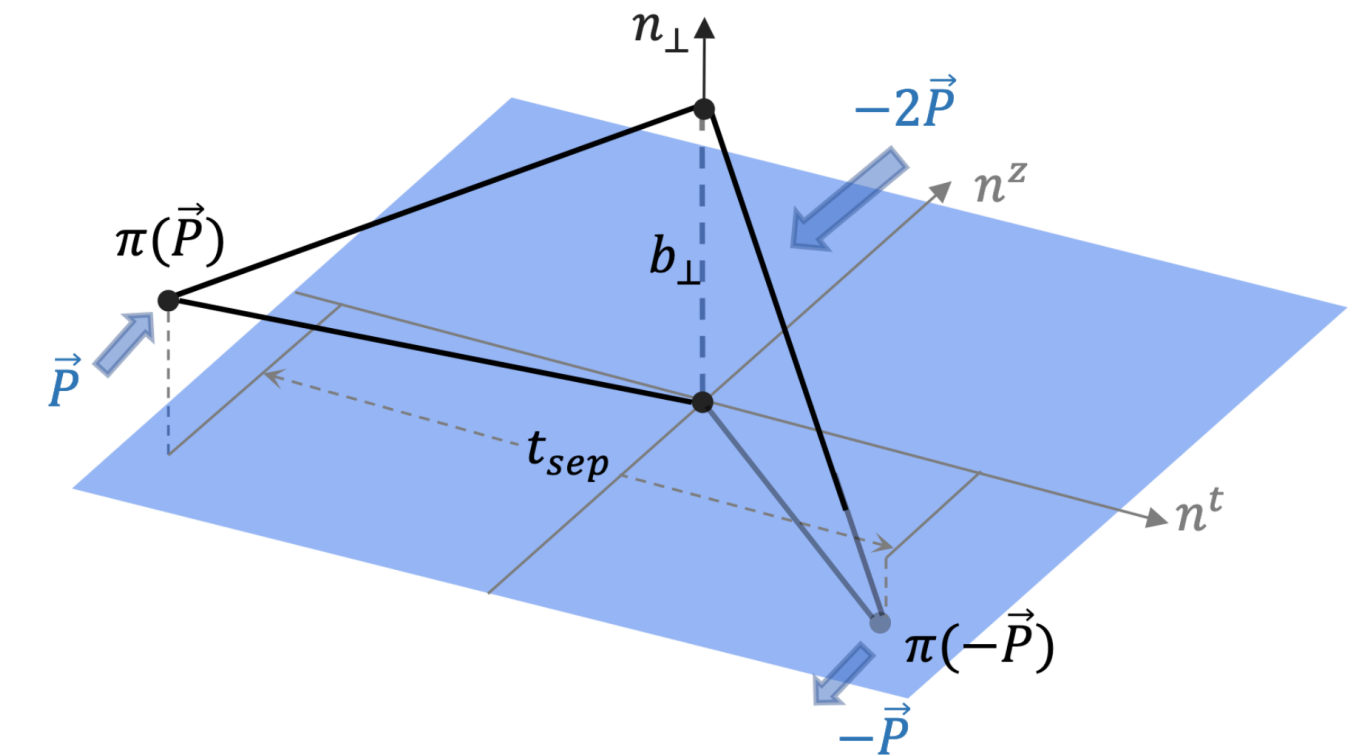
$$\phi(x, b_{\perp}, \dots) = \int_{-\infty}^{\infty} \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \langle 0 | \bar{\psi}(b^{\mu}) W_{\square}(b^{\mu}, 0) \gamma^+ \gamma^5 \psi(0) | P \rangle \text{ is TMD wave function.}$$

$$H_F(x_1, x_2, P^z; \mu) = C_{\text{Sud}}(x_1, x_2, P^z; \mu) \cdot C_{\text{Sud}}(\bar{x}_1, \bar{x}_2, P^z; \mu), \text{ where } C_{\text{Sud}} \text{ is the Sudakov kernel.}$$

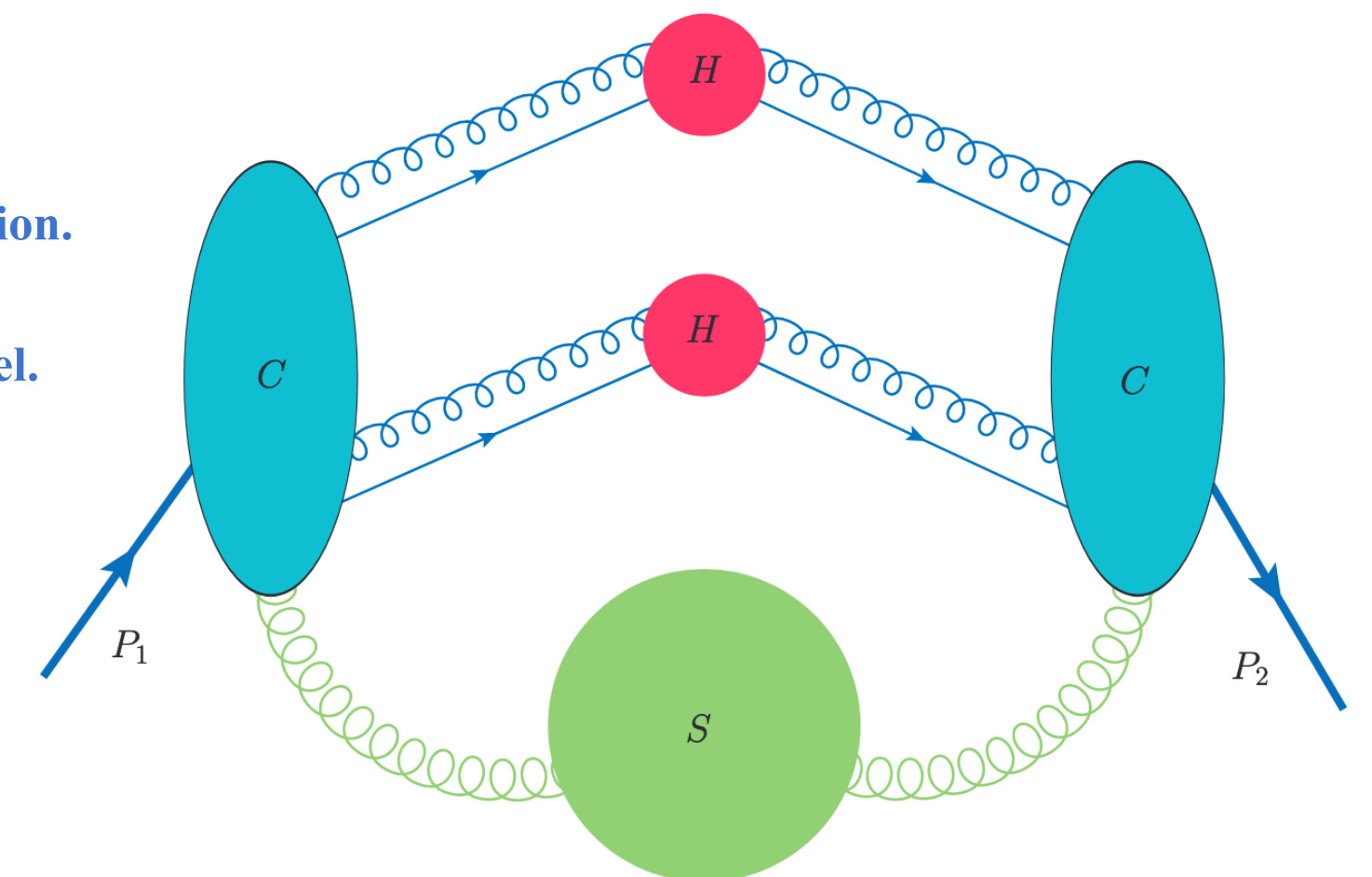
- Therefore, the intrinsic soft function can be extracted via

$$S_I(b_{\perp}; \mu) = \frac{F(b_{\perp}, P^z)}{\int dx_1 dx_2 H_F(x_1, x_2, P^z; \mu) \tilde{\Phi}^{\dagger}(x_1) \tilde{\Phi}(x_2)} \text{ with } \tilde{\Phi}(x) \equiv \frac{\tilde{\phi}_{\Gamma}(x, b_{\perp}, P^z; \mu)}{H_{\phi}(x, \bar{x}, P^z; \mu)}$$

$\tilde{\phi}_{\Gamma}(x, b_{\perp}, P^z; \mu)$ is quasi-TMD wave function.

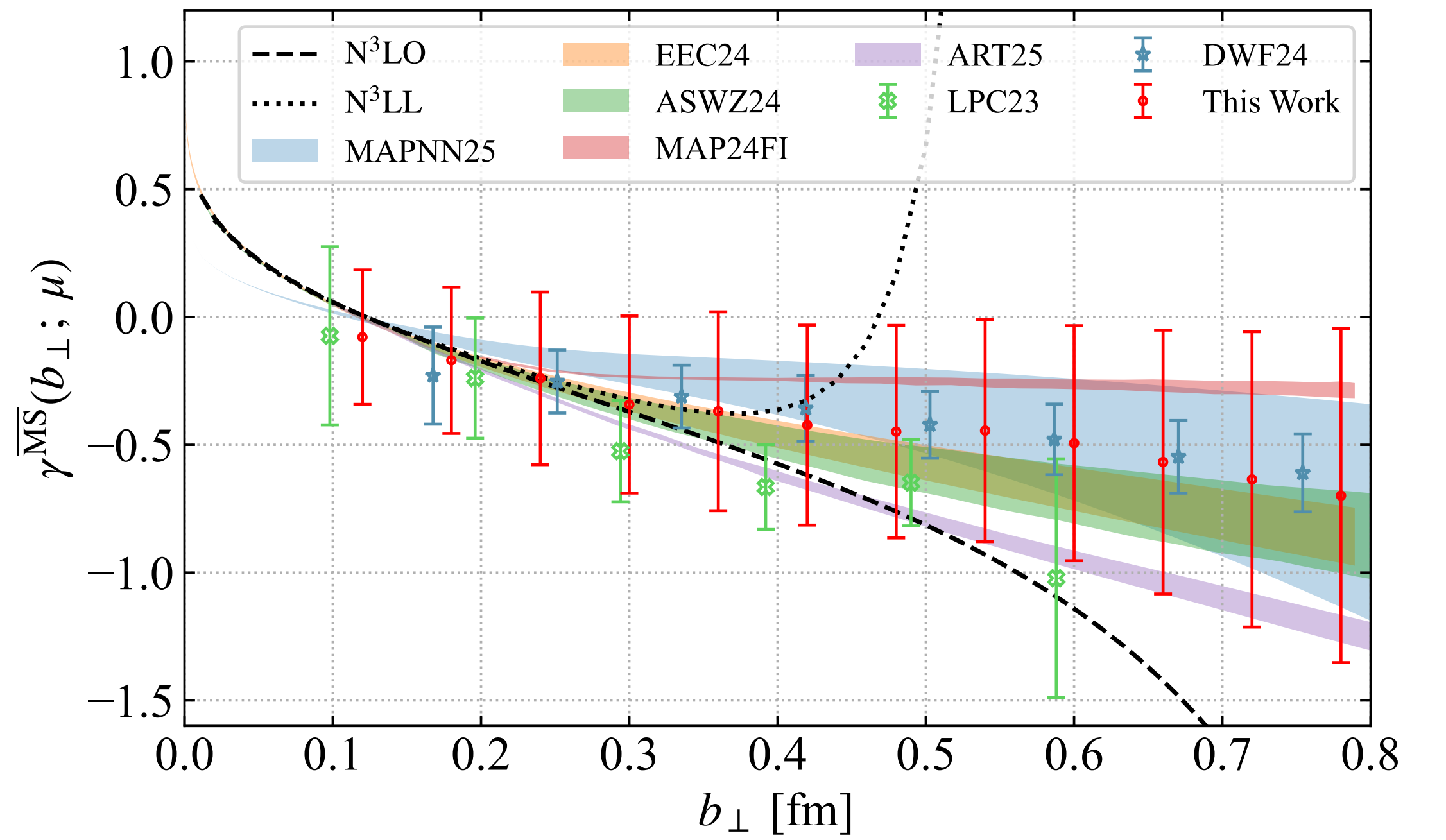
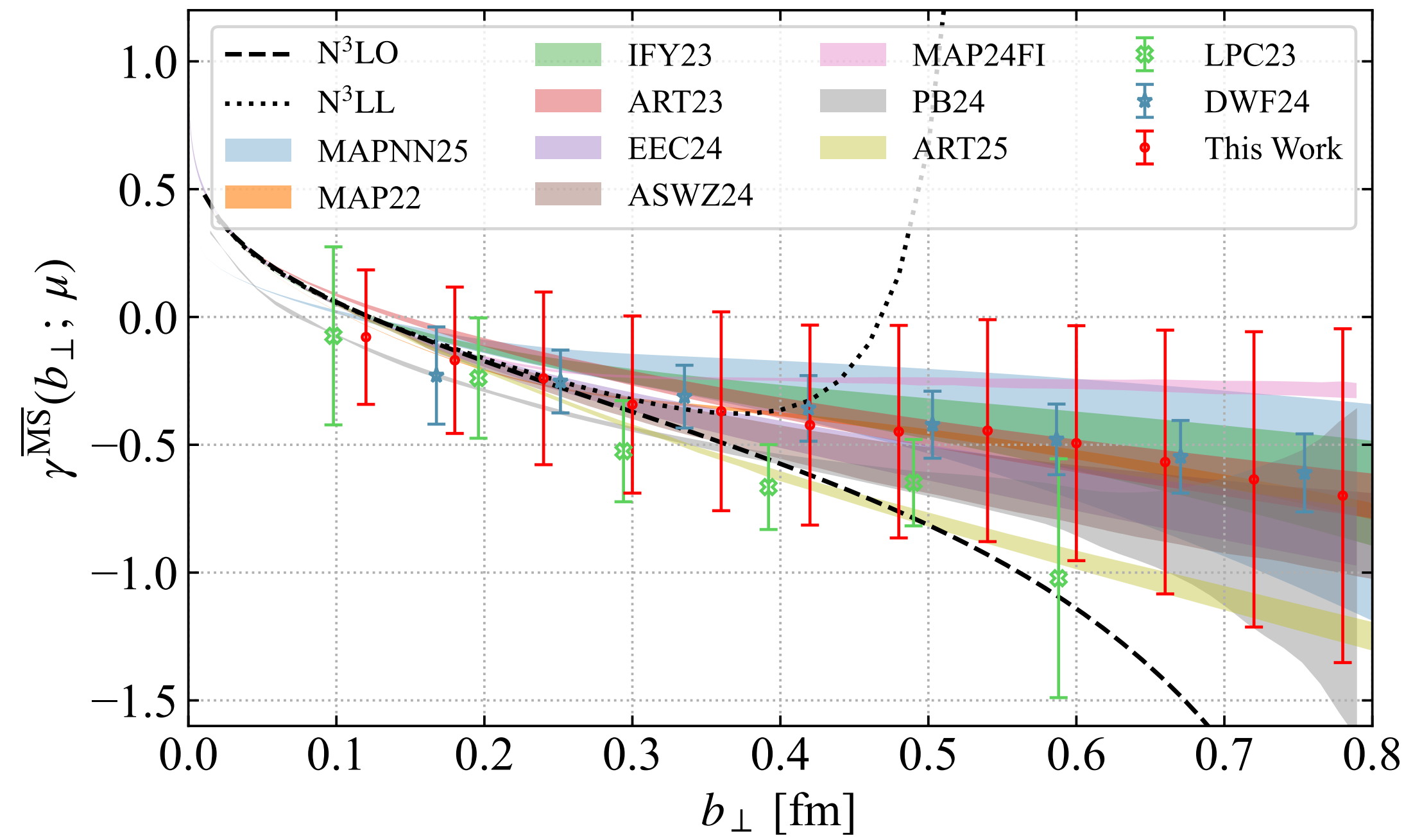


Q. A. Zhang, et al., Phys. Rev. Lett. 125 (2020)

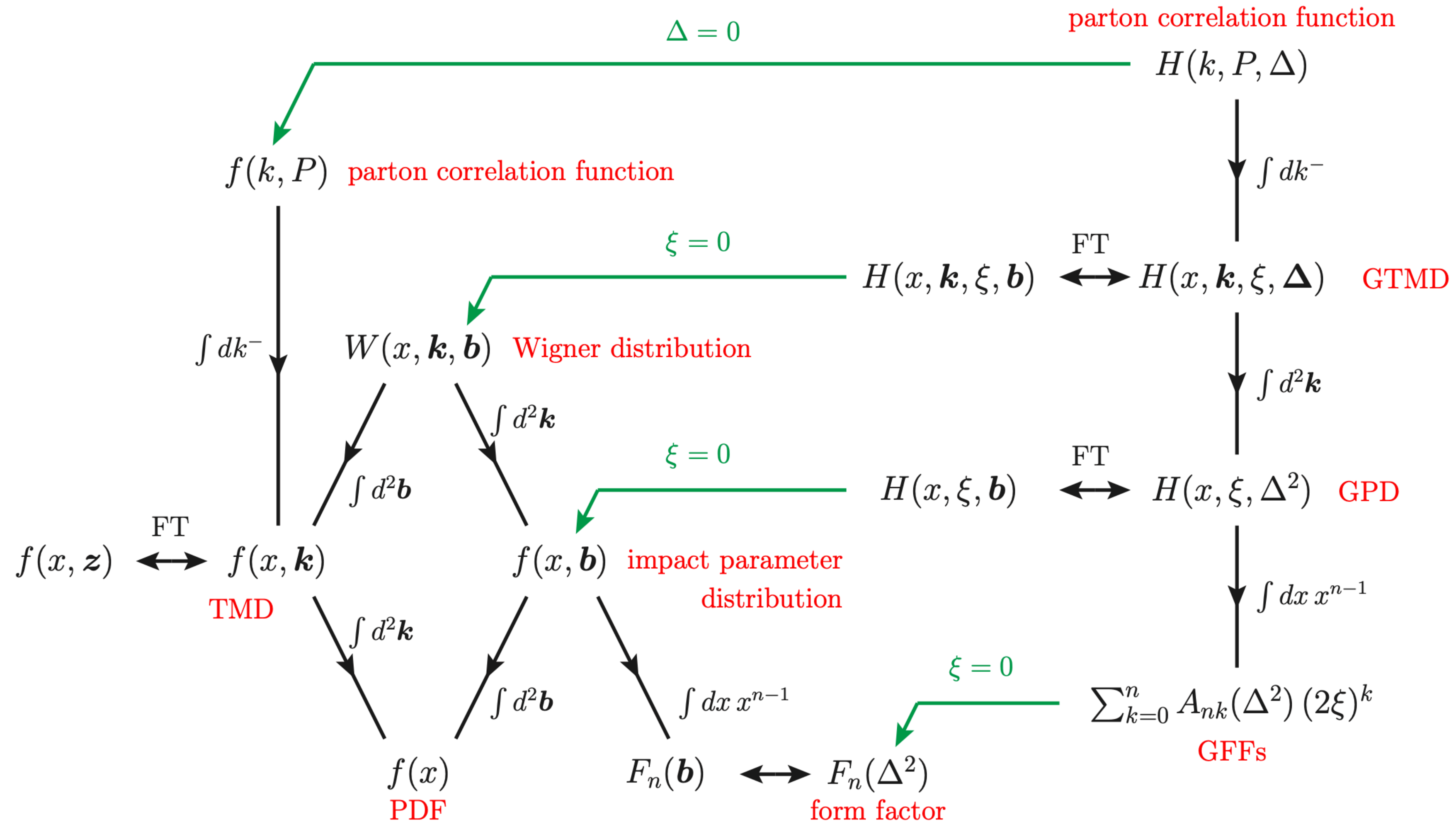


Z. F. Deng, et al., JHEP 09 (2022)

CS Kernel



Partonic Structures



Soft Functions in TMD Factorization

$$\frac{d\sigma}{dQdYd^2\mathbf{q}_T} = \left(\frac{d\sigma^W}{dQdYd^2\mathbf{q}_T} + \frac{d\sigma^Y}{dQdYd^2\mathbf{q}_T} \right) \left[1 + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{Q^2}\right) \right].$$

$$\frac{d\sigma^W}{dQdYd^2\mathbf{q}_T} = \sum_{\text{flavors } i} H_{i\bar{i}}(Q^2, \mu) \int d^2\mathbf{b}_T e^{i\mathbf{b}_T \cdot \mathbf{q}_T} \tilde{f}_{i/p}(x_a, \mathbf{b}_T, \mu, \zeta_a) \tilde{f}_{\bar{i}/p}(x_b, \mathbf{b}_T, \mu, \zeta_b) \quad (2.29a)$$

$$= \sum_{\text{flavors } i} H_{i\bar{i}}(Q^2, \mu) \int d^2\mathbf{b}_T e^{i\mathbf{b}_T \cdot \mathbf{q}_T} \tilde{B}_{i/p}(x_a, \mathbf{b}_T, \mu, \zeta_a/\nu^2) \tilde{B}_{\bar{i}/p}(x_b, \mathbf{b}_T, \mu, \zeta_b/\nu^2) \\ \times \tilde{S}_{n_a n_b}(b_T, \mu, \nu). \quad (2.29b)$$

$$\tilde{f}_{i/p}(x, \mathbf{b}_T, \mu, \zeta) = \tilde{B}_{i/p}(x, \mathbf{b}_T, \mu, \zeta/\nu^2) \sqrt{\tilde{S}_{n_a n_b}(b_T, \mu, \nu)}, \quad (2.32)$$

$$\tilde{B}_{i/p}(x, \mathbf{b}_T, \mu, \zeta/\nu^2) = \lim_{\substack{\epsilon \rightarrow 0 \\ \tau \rightarrow 0}} \tilde{Z}_B^i(b_T, \mu, \nu, \epsilon, \tau, xP^+) \tilde{B}_{i/p}^0(x, \mathbf{b}_T, \epsilon, \tau, xP^+) \\ = \lim_{\substack{\epsilon \rightarrow 0 \\ \tau \rightarrow 0}} \tilde{Z}_B^i(b_T, \mu, \nu, \epsilon, \tau, xP^+) \frac{\tilde{f}_{i/p}^{0(u)}(x, \mathbf{b}_T, \epsilon, \tau, xP^+)}{\tilde{S}_{n_a n_b}^{0\text{subt}}(b_T, \epsilon, \tau)}, \quad (2.34)$$

$$\tilde{S}_{n_a n_b}(b_T, \mu, \nu) = \lim_{\substack{\epsilon \rightarrow 0 \\ \tau \rightarrow 0}} \tilde{Z}_S(b_T, \mu, \nu, \epsilon, \tau) \tilde{S}_{n_a n_b}^0(b_T, \epsilon, \tau). \quad (2.35)$$

$$\tilde{f}_\Gamma(x, b_\perp, P^z; \mu) = H_f(x, P^z; \mu) B(x, b_\perp, xP^+; \mu, \nu) \\ \times S_C^0(b_\perp; \mu, \nu), \quad (3)$$

$$\sqrt{S_I(b_\perp, y_n; \mu)} \cdot \tilde{f}_\Gamma(x, b_\perp, P^z; \mu) \\ = f(x, b_\perp; \mu, \zeta) H_f(x, P^z; \mu) + \text{p.c.}, \quad (5)$$

- Zero-bin subtracted beam function has no soft contribution;
- S_C^0 is the quasi-soft function with zero-bin contribution;
- The intrinsic soft function S_I is the combination of subtracted soft function and the quasi-soft function

Soft Function in TMD Factorization

$$\tilde{S}_{n_a n_b}^0(b_T, \epsilon, \tau) = \frac{1}{N_c} \langle 0 | \text{Tr} [W_{\gg}(b_T)]_{\tau} | 0 \rangle$$

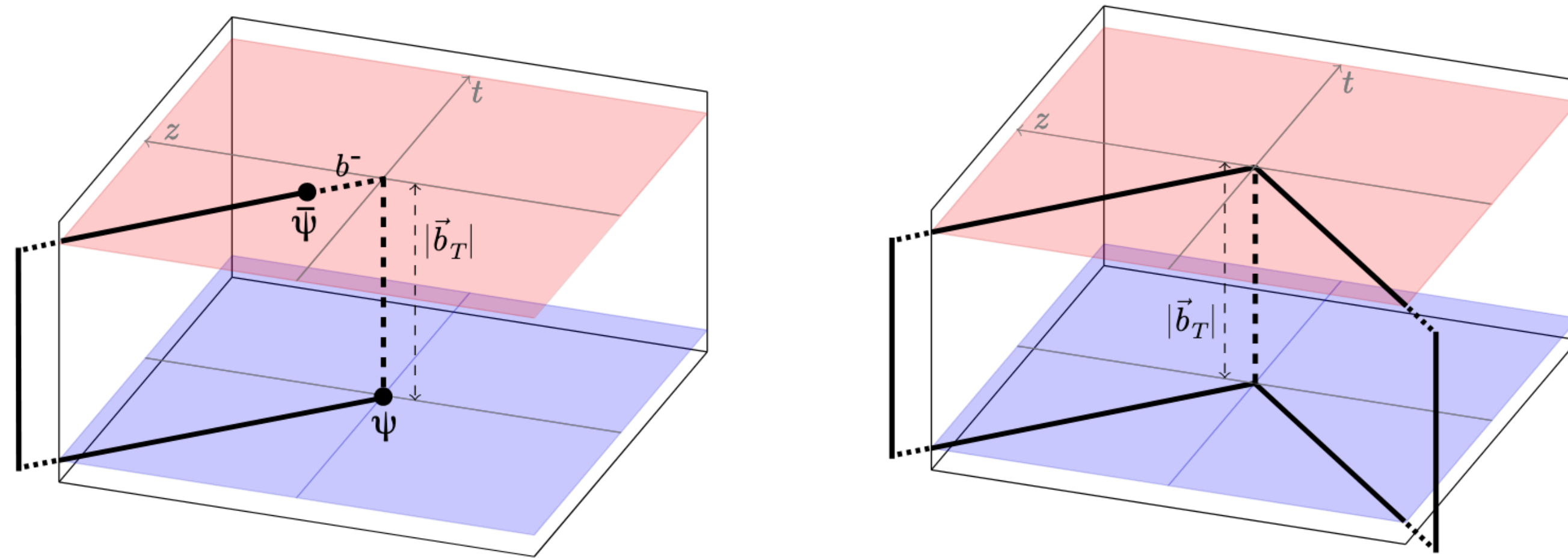
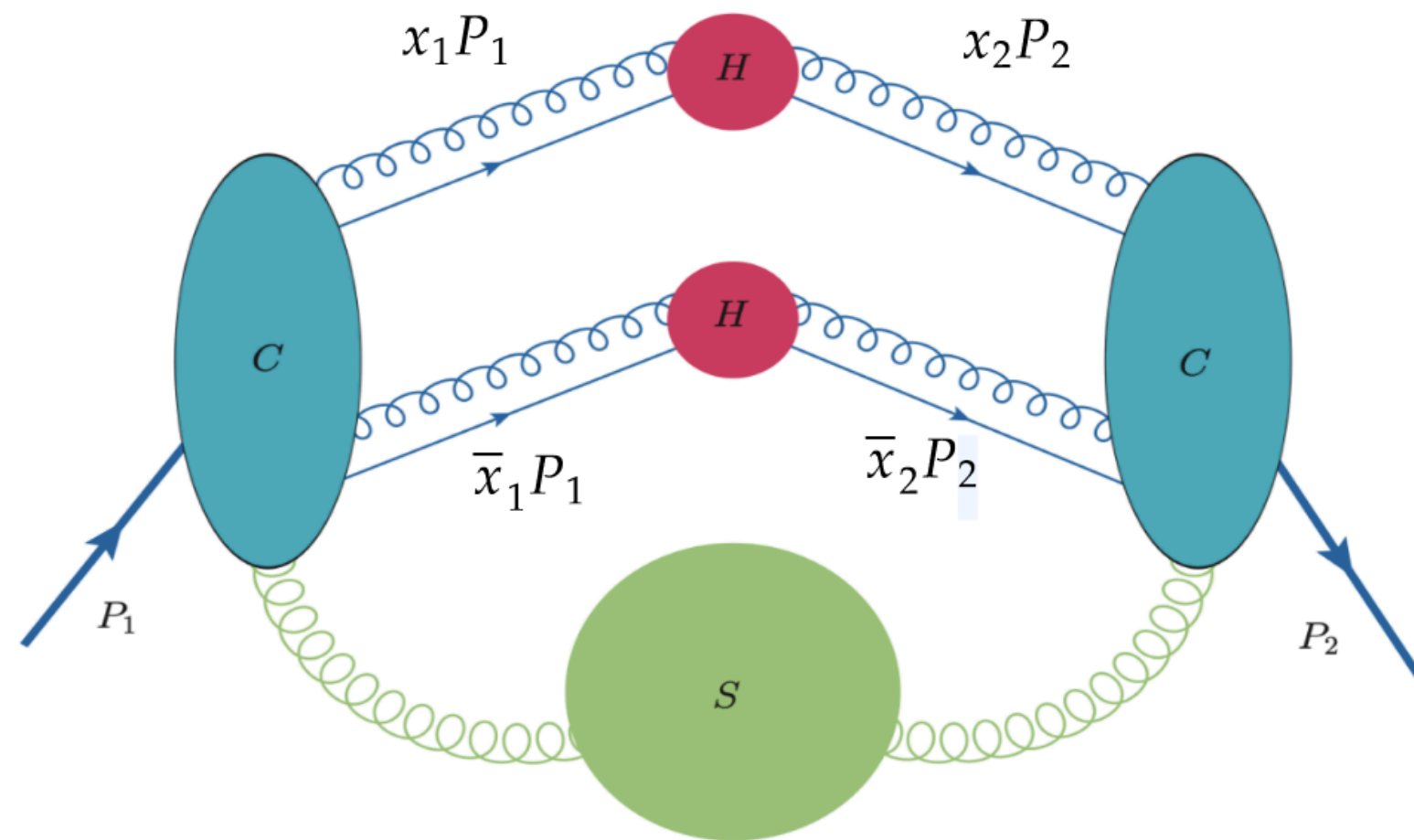


Figure 2.1: Graphs of the Wilson line structure $W_{\square}(b^{\mu}, 0)$ of the unsubtracted TMD PDF $f_{i/p}^{0(u)}$ (left) and of $W_{\gg}(b_T)$ for the soft function $S_{n_a n_b}^0$ (right), defined in Eqs. (2.37) and (2.38). The Wilson lines (solid) extend to infinity in the directions indicated. Adapted from [106].

R. Boussarie, et al., 2304.03302 (2023)

Sudakov Kernel



Z. F. Deng, et al., JHEP 09 (2022)

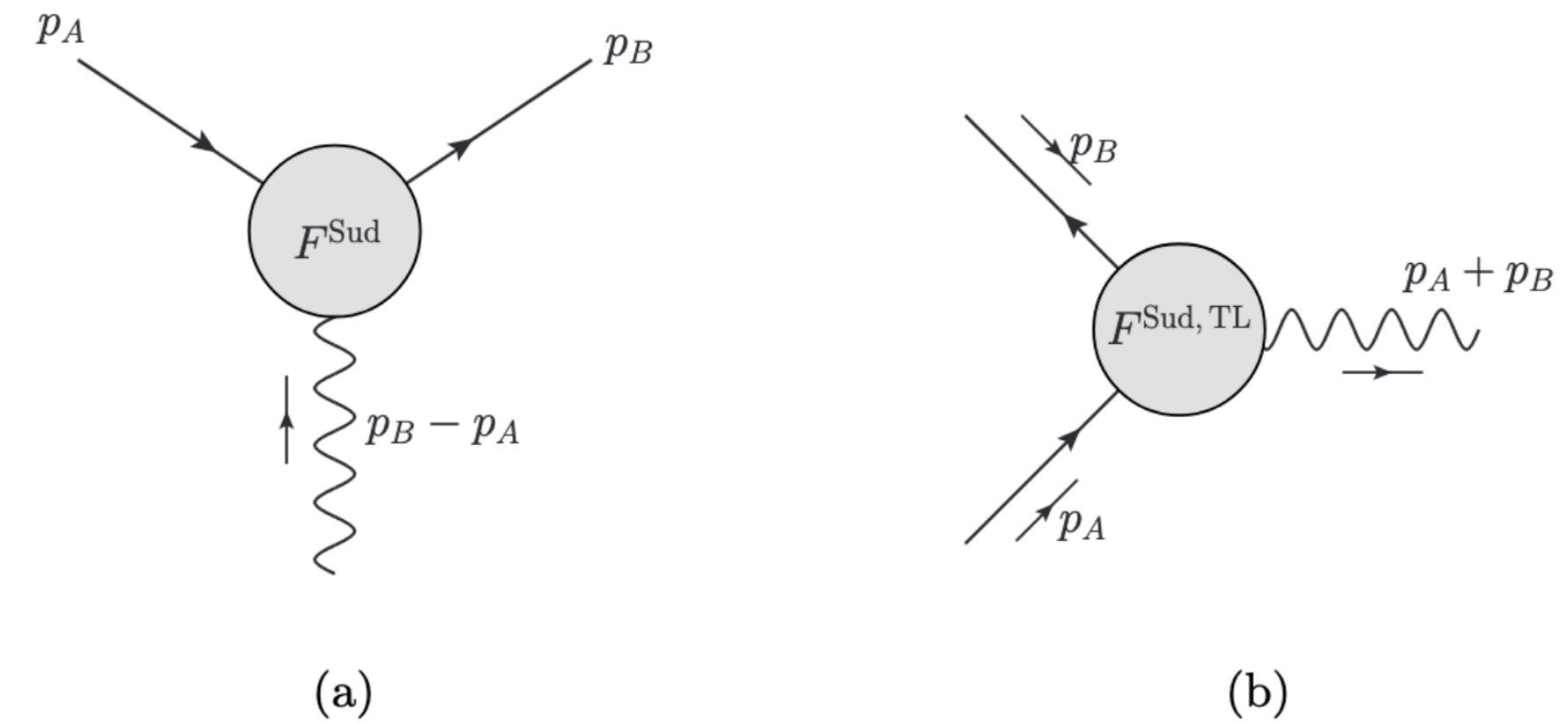


FIG. 1. Graphs for the space-like (a) and time-like (b) quark form factor.

J. Collins and T. C. Rogers, Phys. Rev. D 96 (2017)

$H_F(x_1, x_2, P^z; \mu) = C_{\text{Sud}}(x_1, x_2, P^z; \mu) \cdot C_{\text{Sud}}(\bar{x}_1, \bar{x}_2, P^z; \mu)$, where C_{Sud} is the Sudakov kernel.

$$p_A = (x_1 P_1, 0, 0, x_1 P_1), \quad p_B = (x_2 P_2, 0, 0, -x_2 P_2)$$

$$Q^2 = -(p_B - p_A)^2 = 4x_1 x_2 P_1 P_2$$

$$\bar{Q}^2 = 4\bar{x}_1 \bar{x}_2 P_1 P_2$$

Gauge Fixing in Lattice QCD

Continuous Theory

$$F_{\text{CG}}[A, \Omega] \equiv \frac{1}{2} \sum_{\mu=1}^3 \int d^4x A_{\Omega\mu}^a(x) A_{\Omega}^{\mu a}(x)$$

$$\begin{aligned} \delta F_{\text{CG}}[A, \Omega] &= - \sum_{\mu=1}^3 \int d^4x (D_{\mu ab}^{\Omega} \theta_b) A_{\Omega}^{\mu a} \\ &= - \sum_{\mu=1}^3 \int d^4x (\partial_{\mu} \theta_a - g f^{cab} A_{\Omega\mu}^c \theta_b) A_{\Omega}^{\mu a} \\ &= \sum_{\mu=1}^3 \int d^4x \theta_a (\partial_{\mu} A_{\Omega}^{\mu a}) \end{aligned}$$

$$* A_{\Omega\mu}(x) \equiv \Omega^{\dagger}(x) A_{\mu}(x) \Omega(x) + \frac{i}{g} \Omega^{\dagger}(x) \partial_{\mu} \Omega(x)$$

Lattice Theory

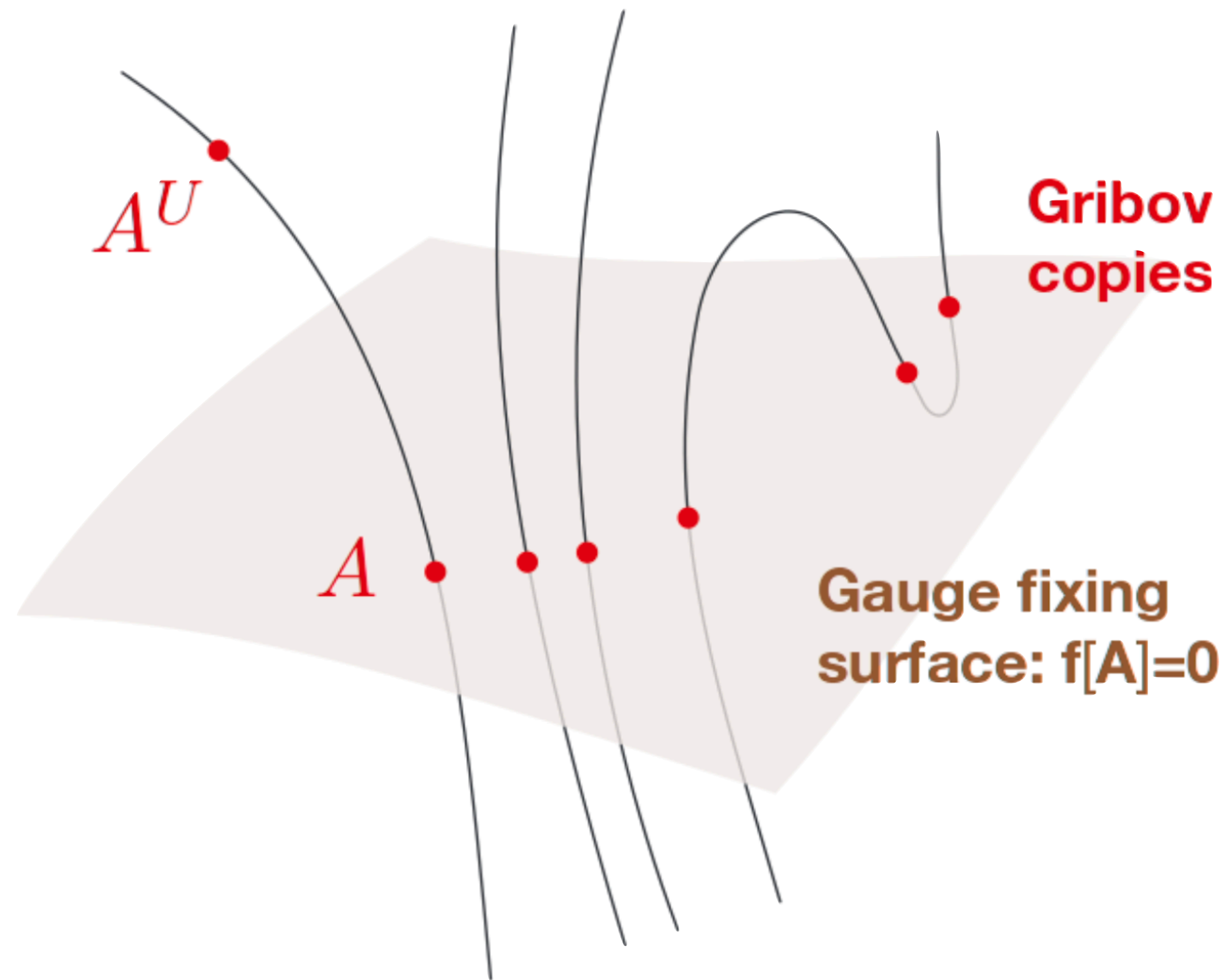
$$F_{\text{CG}}[U, \Omega] \equiv -\Re \left[\text{Tr} \sum_x \sum_{\mu=1}^3 \Omega^{\dagger}(x + \hat{\mu}) U_{\mu}(x) \Omega(x) \right]$$

Find stationary points of the functional value.

Gauge fixing criterion in this work: variation of functional satisfies $\delta F/F < 10^{-8}$.

Gribov Copies

The gauge fixing condition may have many solutions in Lattice QCD.



Ph. D. Thesis of Diego Fiorentini

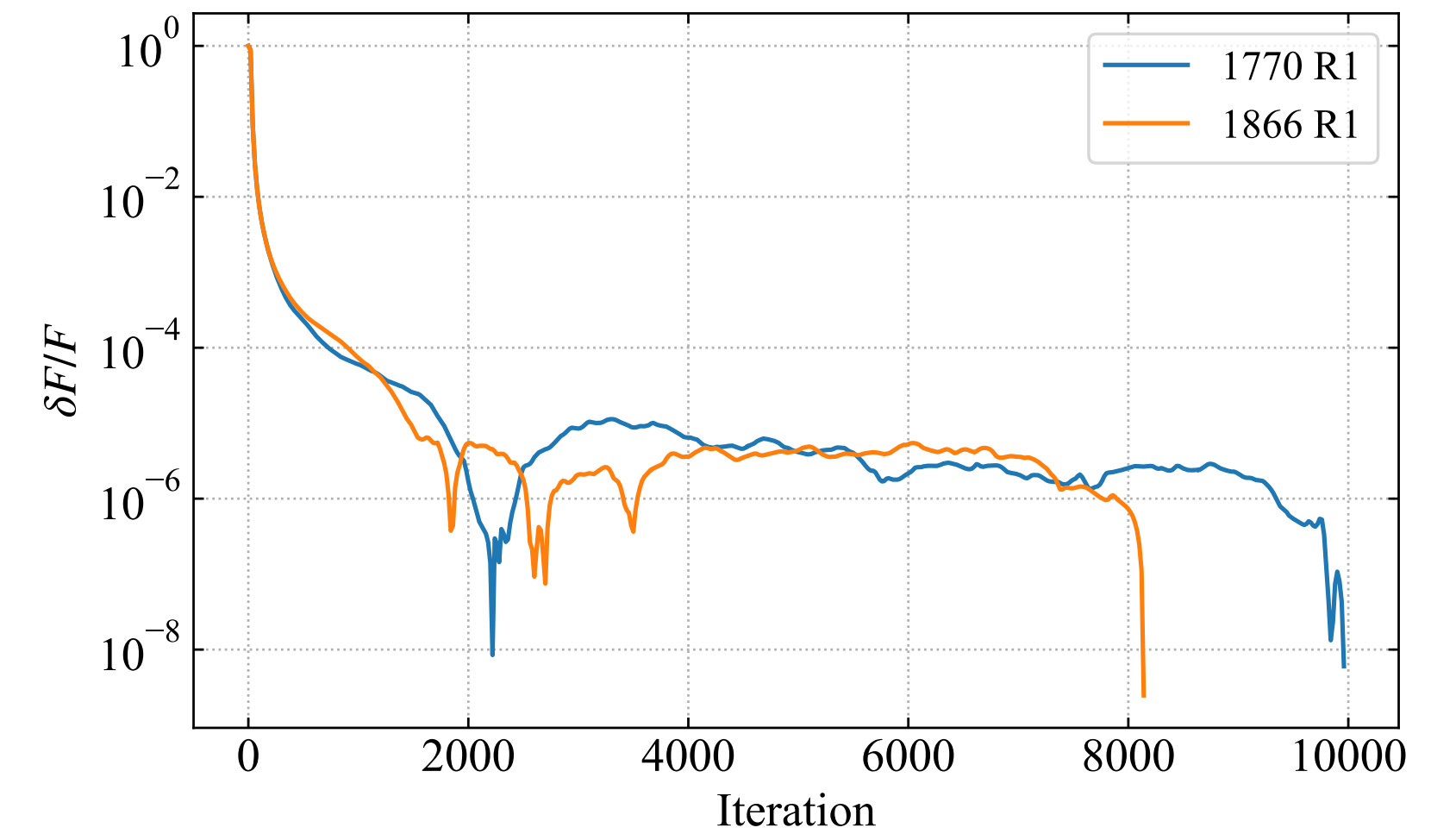
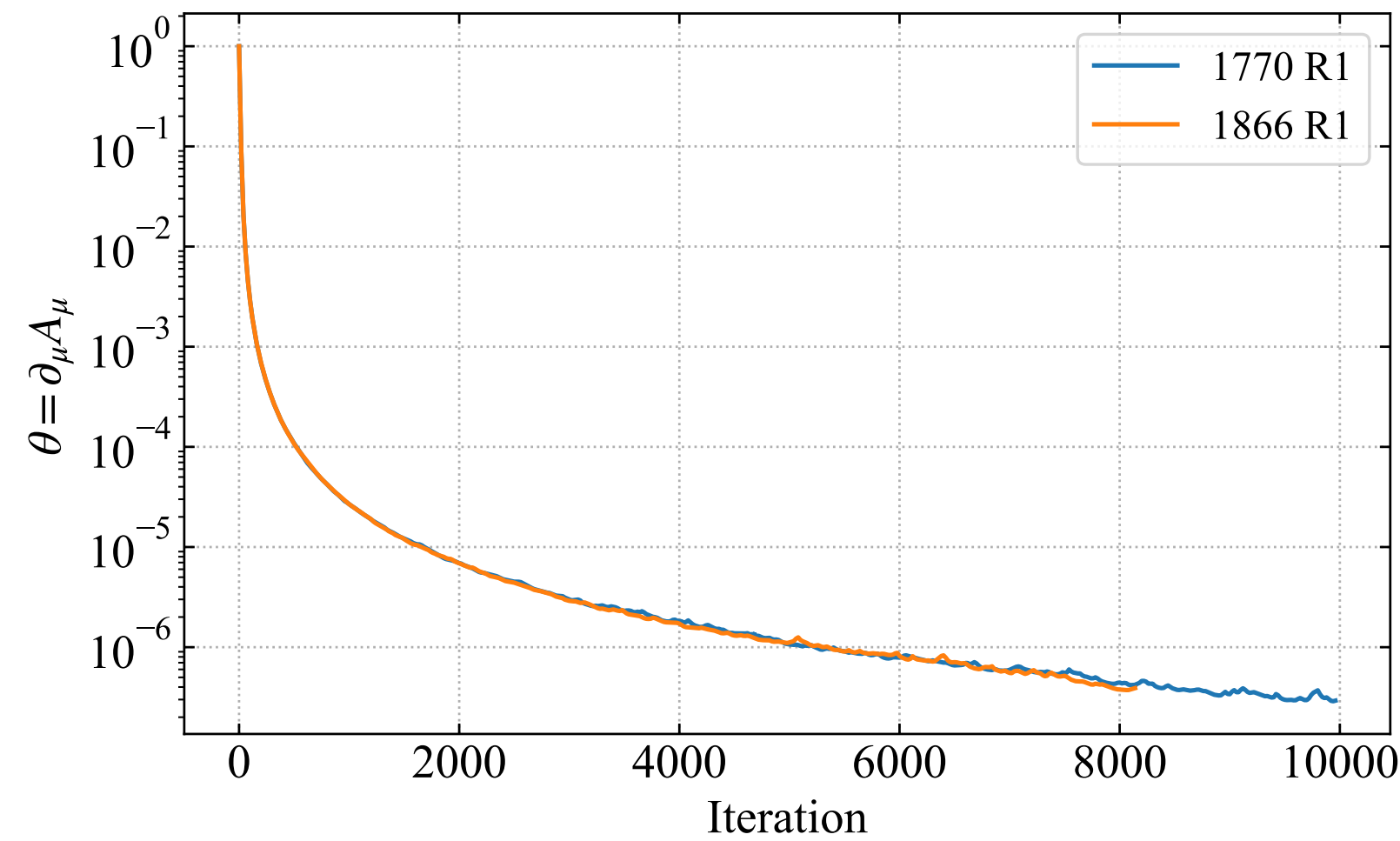
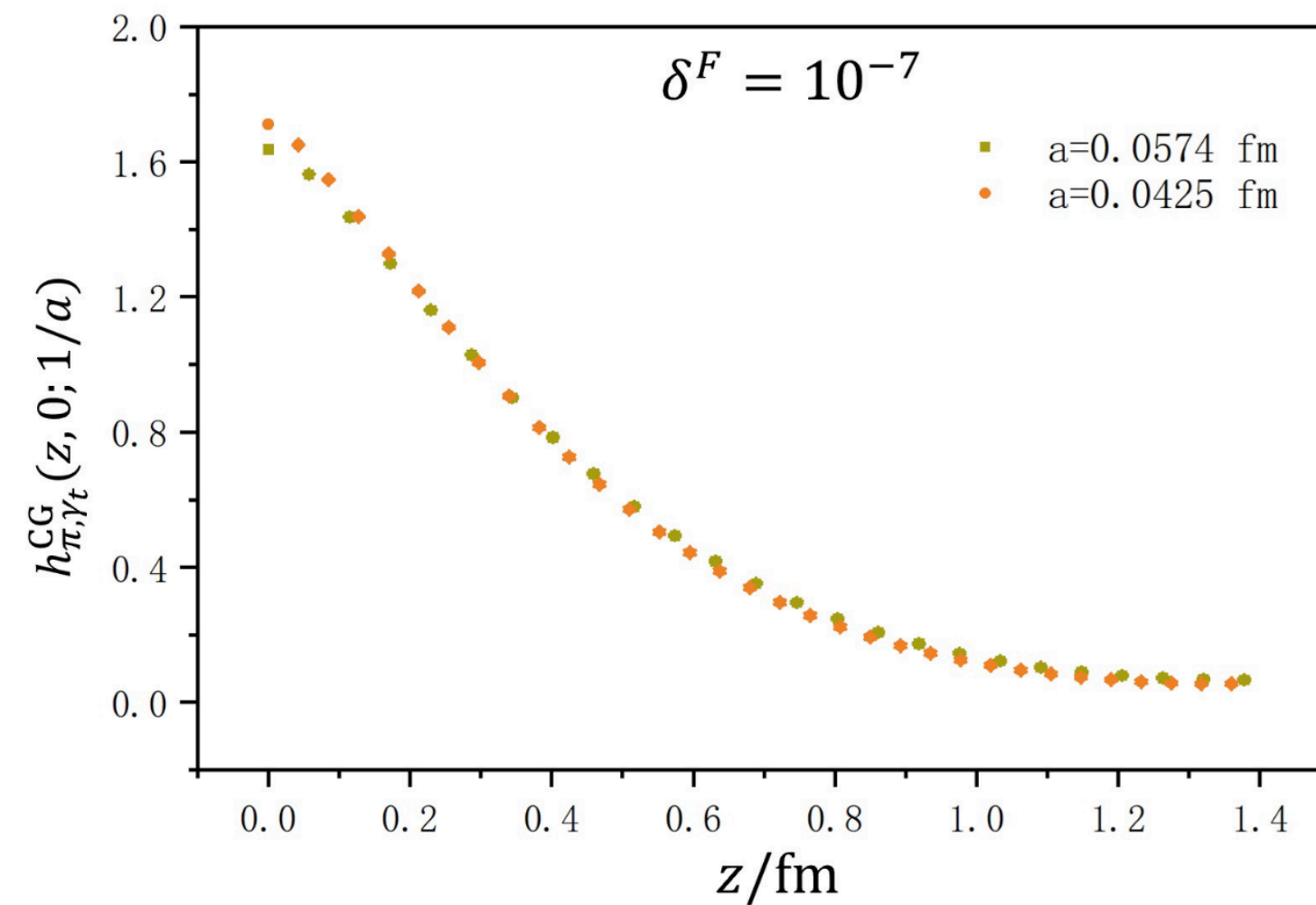
Criteria of Gauge Fixing

- Variation of the functional

$$\delta F/F < 10^{-8}$$

- Residual gradient of the functional

$$\theta^G \equiv \frac{1}{V} \sum_x \theta^G(x) \equiv \frac{1}{V} \sum_x \text{Tr} \left[\Delta^G(x) (\Delta^G)^\dagger(x) \right], \Delta^G(x) \equiv \sum_\mu \left(A_\mu^G(x) - A_\mu^G(x - \hat{\mu}) \right)$$



Preliminary Results with mpi=300

