Wigner-Eckart Theorem

Vector and Tensor operator

Definition of vector operator $\vec{V} = (V_1, V_2, V_3)$

$$U^{-1}(R) V_i U(R) = R_{ij} V_j$$
$$[V_i, J_j] = i\epsilon_{ijk}\hbar V_k$$

replace R by R^{-1}, $R_{ij} \to R_{ji}$ to get

$$U(R) V_i U^{-1}(R) = V_j R_{ji}$$

in which R_{ij} can be seen as the representation matrix elements of $j = 1$.

For more general situation, we have the representation matrix elements $\mathfrak{D}^{(j)}_{mm'}(R)$, so we can define tensor operator.

Definition of tensor operator $T^{(k)}_q$

$$U(R) T^{(k)}_q U^{-1}(R) = \sum_{q'=-k}^{k} T^{(k)}_{q'} \, \mathfrak{D}^{(j)}_{q'q}(R)$$

which is similar to the transform

$$U(R) |j, m\rangle = \sum_{m'} \mathfrak{D}^{(j)}_{m'm}(R) \, |j, m'\rangle$$

We can also check the transform of $T^{(k)}_q |j, m\rangle$

$$U(R)\left(T^{(k)}_q |j, m\rangle\right) = U(R) T^{(k)}_q U^{-1}(R) U(R) |j, m\rangle$$
$$= \sum_{q'} T^{(k)}_{q'} \, \mathfrak{D}^{(j)}_{q'q}(R) \sum_{m'} |j, m'\rangle\langle j, m'| U(R) |j, m\rangle$$
$$= \sum_{q', m'} \left(T^{(k)}_{q'} |j, m'\rangle\right) \mathfrak{D}^{(j)}_{q'q}(R) \mathfrak{D}^{(j)}_{m'm}(R)$$

which is similar to the transform

$$U(R)\left(|j_1, m_1\rangle |j_2, m_2\rangle\right) = \sum_{m_1', m_2'} \mathfrak{D}^{(j_1)}_{m_1'm_1}(R) \mathfrak{D}^{(j_2)}_{m_2'm_2}(R) \left(|j_1, m_1'\rangle |j_2, m_2'\rangle\right)$$

Selection rule

$$\left\langle \alpha', j', m' \left| T^{(k)}_q \right| \alpha, j, m \right\rangle = 0, \text{ unless } \left\{ |k - j| \leqslant j' \leqslant |k + j| \text{ and } m' = m + q \right\}$$

in which α and α' are quantum numbers apart from angular momentum.

Wigner-Eckart Theorem

Because $T_q^{(k)}$ has the same transformation as $|j, m\rangle$, we can use CG coefficients to combine two spherical tensor operators to a new spherical tensor operator

$$A_{m_1}^{(j_1)} + B_{m_2}^{(j_2)} \to T_m^{(j)}$$

With the definition $\mathfrak{D}_{m'm}^{(j)}(R) = \langle j, m' | U(R) | j, m\rangle$, we can rewrite the transformation of $T_q^{(k)}$ to get

$$U(\hat{n}, \theta) T_q^{(k)} U^{-1}(\hat{n}, \theta) = \sum_{q'} T_{q'}^{(k)} \langle k, q' | U(\hat{n}, \theta) | k, q\rangle$$

take infinitesimal rotation $\theta = \epsilon$, we got

$$\left[\vec{J} \cdot \hat{n}, T_q^{(k)}\right] = \sum_{q'} T_{q'}^{(k)} \left\langle k, q' \middle| \vec{J} \cdot \hat{n} \middle| k, q\right\rangle$$

replace $\vec{J} \cdot \hat{n}$ by $J_\pm$, we got

$$\left[J_\pm, T_q^{(k)}\right] = \sum_{q'} T_{q'}^{(k)} \langle k, q' | J_\pm | k, q\rangle = \hbar\sqrt{k(k+1) - q(q \pm 1)}\, T_{q\pm 1}^{(k)}$$

replace $\vec{J} \cdot \hat{n}$ by J_z, we got

$$\left[J_z, T_q^{(k)}\right] = \sum_{q'} T_{q'}^{(k)} \langle k, q' | J_z | k, q\rangle = \hbar q\, T_q^{(k)}$$

Then we can prove the **Wigner-Eckart theorem**

$$\left\langle \alpha', j, m \middle| T_{m_1}^{(j_1)} \middle| \alpha, j_2, m_2 \right\rangle = C_{j_1, j_2}(j, m; m_1, m_2) \cdot \langle \alpha', j || T^{(j_1)} || \alpha, j_2\rangle$$

in which $C_{j_1, j_2}(j, m; m_1, m_2) = \langle j_1 j_2; m_1 m_2 \mid j_1 j_2; j, m\rangle$.

To prove it, we just need to prove that the matrix elements $\left\langle \alpha', j, m \middle| T_{m_1}^{(j_1)} \middle| \alpha, j_2, m_2 \right\rangle$ satisfy the same recursion relation as CG coefficients

$$\begin{aligned}
&\sqrt{(j \mp m)(j \pm m + 1)}\, \langle j_1 j_2; m_1 m_2 \mid j_1 j_2; j, m \pm 1\rangle \\
&= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)}\, \langle j_1 j_2; m_1 \mp 1, m_2 \mid j_1 j_2; jm\rangle \\
&+ \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)}\, \langle j_1 j_2; m_1, m_2 \mp 1 \mid j_1 j_2; jm\rangle
\end{aligned}$$

With

$$\left[J_\pm, T_q^{(k)}\right] = \hbar\sqrt{k(k+1) - q(q \pm 1)}\, T_{q\pm 1}^{(k)}$$

we have

$$\left\langle \alpha', j, m \middle| \left[J_\pm, T^{(j_1)}_{m_1} \right] \middle| \alpha, j_2, m_2 \right\rangle = \hbar\sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)} \left\langle \alpha', j, m \middle| T^{(k)}_{q\pm1} \middle| \alpha, j_2, m_2 \right\rangle$$

the recursion relation of the matrix elements $\left\langle \alpha', j, m \middle| T^{(j_1)}_{m_1} \middle| \alpha, j_2, m_2 \right\rangle$ can be deduced.

<u>Some inferences</u>

For a scalar operator S, we have

$$\langle \alpha', j', m' | S | \alpha, j, m \rangle = \delta_{j,j'} \delta_{m,m'} \cdot \langle \alpha', j' || S || \alpha, j \rangle$$

For a vector operator $\vec{V}$, we have

$$\langle \alpha', j, m' | V_q | \alpha, j, m \rangle = \frac{\langle \alpha', j, m | \vec{J} \cdot \vec{V} | \alpha, j, m \rangle}{j(j+1)\hbar^2} \langle j, m' | J_q | j, m \rangle$$