Definition in textbook

According to Foundations of Perturbative QCD of Collins, we have the plus function defined as below

$$\int_0^1 d\xi \left(\frac{1}{1-\xi}\right)_+ T(\xi) \equiv \int_0^1 d\xi \frac{T(\xi)-T(1)}{1-\xi} \tag{1}$$

The plus function can be effectively written as

$$\left(\frac{1}{1-\xi}\right)_+ = \frac{1}{1-\xi} - \delta(1-\xi)\int_0^1 d\xi' \frac{1}{1-\xi'} \tag{2}$$

Plus function in LaMET matching

In the LaMET matching, we have

$$f(x) = \int_0^1 dy\, C(x,y)_+ \tilde{f}(y) = \int_0^1 dy\, [C(x,y)\tilde{f}(y) - C(y,x)\tilde{f}(x)] \tag{3}$$

this can be verified by assuming $C(x,y) = \frac{1}{1-x/y} = \frac{1}{1-\xi}$, where $\xi = \frac{x}{y}$ and $dy = -x/\xi^2\, d\xi$.

Using Eq. (1), the LHS of Eq. (3) becomes

$$\int_0^1 dy\, C(x,y)_+ \tilde{f}(y) = \int_0^1 d\xi \frac{-x}{\xi^2}\left(\frac{1}{1-\xi}\right)_+ \tilde{f}(x/\xi) = \int_0^1 d\xi \frac{1}{1-\xi}\left(\frac{-x}{\xi^2}\tilde{f}(x/\xi) - (-x\tilde{f}(x))\right)$$

Note that the second term in the parentheses is independent on the integral variable, so we can rewrite it as

$$\int_0^1 dy\, C(x,y)_+ \tilde{f}(y) = \int_0^1 d\xi \frac{1}{1-\xi}\frac{-x}{\xi^2}\tilde{f}(x/\xi) + x\tilde{f}(x)\int_0^1 d\xi' \frac{1}{1-\xi'}$$

change the variable ξ back to x and y, we got

$$\begin{aligned}\int_0^1 dy\, C(x,y)_+ \tilde{f}(y) &= \int_0^1 dy \frac{-x}{y^2}\frac{-xy^2}{x^2}\frac{1}{1-x/y}\tilde{f}(y) + x\tilde{f}(x)\int_0^1 d\xi' \frac{1}{1-\xi'} \\ &= \int_0^1 dy \frac{\tilde{f}(y)}{1-x/y} + \tilde{f}(x)\int_0^1 d\xi' \frac{x}{1-\xi'}\end{aligned}$$

if $\xi' = y/x$, then we got

$$\int_0^1 dy\, C(x,y)_+ \tilde{f}(y) = \int_0^1 dy \frac{\tilde{f}(y)}{1-x/y} + \tilde{f}(x)\int_0^1 dy \frac{1}{1-y/x} = \int_0^1 dy\, [C(x,y)\tilde{f}(y) - C(y,x)\tilde{f}(x)]$$

The Eq. (3) is verified.

Numerical implementation of plus function

Similar to Eq. (2), we can implement the plus function in another way

$$C(x,y)_+ = C(x,y) - \delta(x-y)\int_0^1 dz\, C(z,y)$$

then

$$f(x) = \int_0^1 dy\, C(x,y)_+ \tilde{f}(y) = \int_0^1 dy\, C(x,y)\, \tilde{f}(y) - \int_0^1 dy\, \delta(x-y)\int_0^1 dz\, C(z,y)\tilde{f}(y)$$

$$= \int_0^1 dy\, C(x,y)\, \tilde{f}(y) - \int_0^1 dz\, C(z,x) \tilde{f}(x)$$

which is consistent with the definition above.

Therefore, we have

$$\int_0^1 dx\, C(x,y)_+ = \int_0^1 dx\, C(x,y) - \int_0^1 dz\, C(z,y) = 0$$

$$\int_0^1 dy\, C(x,y)_+ = \int_0^1 dy\, C(x,y) - \int_0^1 dz\, C(z,x)$$