Discrete symmetry

Symmetries in classical physics and quantum physics

In classical physics, if the system has the symmetry under the translate

$$q_i \to q_i + \delta q_i$$

it means $\frac{\partial L}{\partial q_i} = \dot{p}_i = 0$, so p_i is a conserved quantity.

In quantum physics, if the system has the symmetry under the translation

$$U = 1 - \frac{i\epsilon}{\hbar}G$$

it means $U^\dagger H U = H$, so $[G, H] = \frac{dG}{dt} = 0$, that means G is a constant of the motion.
Another way to understand is that $[G, H] = 0$ means the eigenstate of G will always be the eigenstate with the same eigenvalue at later time.

Degeneracy

If the system has a symmetry $[U(\lambda), H] = 0$, suppose $H|n\rangle = E_n|n\rangle$, then we have

$$H\big(U(\lambda)|n\rangle\big) = E_n\big(U(\lambda)|n\rangle\big)$$

so the eigenstate of H is degenerate.

Example: Consider an atomic electron with potential $V(r) + V_{LS}(r)\vec{L}\cdot\vec{S}$, then $[\vec{J}, H] = \left[\vec{J}^2, H\right] = 0$. It means the system is rotationally invariant. Because $|n, l, m\rangle$ is $(2j+1)$-fold, here we expect a $(2j+1)$ degeneracy as well. If there is an external electric or magnetic field, the rotation symmetry will be broken, and the $(2j+1)$ degeneracy is no longer expected.

Coulomb Potential

We know that coulomb potential has the form $V(r) \sim 1/r$, which maintains the orientationof the major axis of the ellipse. A small deviation from a $1/r$ potential leads to precession of this axis, so $1/r$ potential actually has an extra constant of motion / extra symmetry, which corresponds to $SO(4)$ group with six generators.

Space Inversion

Suppose parity operator is π, we require the expectation value of $\vec{x}$ taken with respect to the space-inverted state to be opposite in sign

$$\langle\alpha|\pi^\dagger\vec{x}\pi|\alpha\rangle = -\langle\alpha|\vec{x}|\alpha\rangle$$
$$\vec{x}\pi = -\pi\vec{x}$$

So, we have $\pi|\vec{x}\rangle = e^{i\delta}|-\vec{x}\rangle$, choose $e^{i\delta} = 1$, we have $\pi|\vec{x}\rangle = |-\vec{x}\rangle$.

Therefore, $\pi^2|x\rangle = |x\rangle$ and $\pi = \pi^{-1}$. Also, $\langle\vec{x}|\pi|\vec{x}\rangle$ should be a real number, so $\pi = \pi^\dagger = \pi^{-1}$. Parity operator is unitary and Hermitian with eigenvalue ± 1.

Consider the commutation relation between π and $\vec{x}$, we have $\{\pi, \vec{x}\} = 0$. This relation is also correct to momentum $\vec{p} = m\frac{d\vec{x}}{dt}$, so we have $\{\pi, \vec{p}\} = 0$. These operators are called vector operator.

Operators under parity

- Vector:

$$\{\pi, \vec{x}\} = \{\pi, \vec{p}\} = 0$$

- Pseudovector:

$$[\pi, \vec{L}] = [\pi, \vec{S}] = [\pi, \vec{J}] = 0$$

- Scalar:

$$[\pi, \vec{L}\cdot\vec{S}] = [\pi, \vec{x}\cdot\vec{p}] = 0$$

- Pseudoscalar:

$$\{\pi, \vec{S}\cdot\vec{x}\} = 0$$

Wave functions under parity

Considering the eigenvalue of π must be ± 1, we have

$$\langle\vec{x}|\pi|\alpha\rangle = \pm\langle\vec{x}|\alpha\rangle = \langle -\vec{x}|\alpha\rangle$$
$$\psi(-\vec{x}) = \pm\psi(\vec{x})\begin{cases}\text{even parity}\\ \text{odd parity}\end{cases}$$

The eigenstate of $\vec{L}^2$ and L_z should also be the eigenstate of parity because of the commutation relation between $\vec{L}$ and π. The wave function of the eigenstates are

$$\langle\vec{x}|\alpha, l, m\rangle = R_\alpha(r)\cdot Y_l^m(\theta, \phi)$$

When $\vec{x} \to -\vec{x}$, we have parameters change as $\vec{r} \to \vec{r}$, $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$, with the explict form

$$Y_l^m(\theta, \phi) = (-1)^m\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$$

we have $Y_l^m \to (-1)^l Y_l^m$ under parity.

Actually we can just look at $|l, m = 0\rangle$, because $[\pi, L_\pm] = 0$ and $|l, m\rangle$ will have the same parity as $|l, m = 0\rangle$.

Time-Reversal Symmetry

Classical Newton mechanics has the time-reversal symmetry because if $\vec{x}(t)$ is a solution to

$$m\ddot{\vec{x}} = -\nabla V(\vec{x})$$

then $\vec{x}(-t)$ is also a possible solution with the same $V(\vec{x})$.

Anti-unitary

If we have a transformation

$$|\alpha\rangle \to \Theta|\alpha\rangle,\ |\beta\rangle \to \Theta|\beta\rangle$$
$$\langle\beta|\Theta^\dagger\Theta|\alpha\rangle = \langle\beta|\alpha\rangle^*$$

then this transformation is anti-unitary, and it can be written as $\Theta = UK$, where K is the complex conjugate operator that forms the complex conjugate of any coefficient that multiplies a ket.

Because of Wigner's theorem, any symmetric transformation operator can only be unitary or anti-unitary.

Time-reversal operator

Firstly think about space inversion, it is easy to understand that if the system takes space inversion then translates δx in $+x$-direction, it will be the same as the original system translates δx in $-x$-direction, then takes space inversion. Express in formula as

$$\left(1 - \frac{ip_x}{\hbar}\delta x\right)\pi|\alpha\rangle = \pi\left(1 - \frac{ip_x}{\hbar}(-\delta x)\right)|\alpha\rangle$$

Similarly, we require the same relation on time-reversal as

$$\left(1 - \frac{iH}{\hbar}\delta t\right)\Theta|\alpha\rangle = \Theta\left(1 - \frac{iH}{\hbar}(-\delta t)\right)|\alpha\rangle$$
$$-iH\Theta|\alpha\rangle = \Theta iH|\alpha\rangle$$

If we throw away the i on both sides, we will get

$$-H\Theta = \Theta H$$

which gives negative energy. Therefore, we set Θ as an anti-unitary operator that satisfies

$$H\Theta = \Theta H$$

Considering the properties of anti-unitary operators, $\langle\beta|\Theta|\alpha\rangle$ will be understood as $\langle\beta|\big(\Theta|\alpha\rangle\big)$.

Operators under time-reversal

$$\Theta\vec{x}\Theta^{-1} = \vec{x}$$
$$\Theta\vec{p}\Theta^{-1} = -\vec{p}$$
$$\Theta\vec{J}\Theta^{-1} = -\vec{J}$$
$$\Theta\vec{\sigma}\Theta^{-1} = -\vec{\sigma},\ \Theta = (-i\sigma_2)K$$
$$\Theta\vec{E}\Theta^{-1} = \vec{E}$$
$$\Theta\vec{B}\Theta^{-1} = -\vec{B}$$

Because $(i\sigma_2)^2 = -I$, the half spin system will not go back but get an extra minus sign after double time-reversal.

Wave functions under time-reversal

$$|\alpha\rangle = \int d^3x \, \langle\vec{x}|\alpha\rangle \cdot |\vec{x}\rangle$$
$$\Theta|\alpha\rangle = \int d^3x \, \langle\vec{x}|\alpha\rangle^* \cdot \Theta|\vec{x}\rangle$$

choose $\Theta|\vec{x}\rangle = |\vec{x}\rangle$, then we got

$$\Theta|\alpha\rangle = \int d^3x \, \langle\vec{x}|\alpha\rangle^* \cdot |\vec{x}\rangle$$

so the wave function transform under time-reversal as

$$\psi(\vec{x}) \rightarrow \psi^*(\vec{x})$$
$$Y_l^m(\theta,\phi) \rightarrow {Y_l^m}^*(\theta,\phi) = (-1)^m Y_l^{-m}(\theta,\phi)$$