Beta Function of QED

Key points

- RG evolution and equation
- The Callan-Symanzik Equation
- 1-loop renormalization of QED
- Another way to get the β-function

Details

RG evolution and equation

Two parameters are introduced during renormalization: arbitrary parameter μ and renormalization scale M.

- The arbitrary parameter μ is introduced in the Dimensional Regularization to balance the dimension of the coupling g.
- The renormalization scale M is the scale that we set the renormalization condition, in on-shell condition, it is the scale that you do experimental measurements to get physical values.

Renormalization is changing variables but leaves Lagrangian invariant, so

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 - \frac{Z_\lambda\lambda\mu^\epsilon}{4!}\phi_0^4$$

where

$$\phi_0 = Z_\phi^{1/2}\phi(M)$$
$$m_0 = Z_m^{1/2}Z_\phi^{-1/2}m(M)$$
$$\lambda_0 = Z_\phi^{-2}Z_\lambda\lambda(M)\mu^\epsilon$$

It can be found that those bare quantities do not depend on μ nor M, so the choice of μ and M does not affect the field theory and its prediction. [Next we will use $\overline{MS}$ scheme, so there is no M dependence in renormalization parameters. In contrast, if we use on-shell scheme, we will have M dependence.]

Therefore, to get the RG equation of coupling, take derivative of μ on λ_0, we got

$$\ln\lambda_0 = \ln\left(Z_\phi^{-2}Z_\lambda\right) + \ln\lambda + \epsilon\ln\mu$$
$$0 = \frac{1}{Z_\phi^{-2}Z_\lambda}\frac{d}{d\ln\mu}\left(Z_\phi^{-2}Z_\lambda\right) + \frac{1}{\lambda}\frac{d}{d\ln\mu}\lambda + \epsilon$$

After the 1-loop calculation, we got the renormalization constants Z, then the RG equation above can be solved to get the evolution of the coupling constant.

For example in ϕ^4 theory with $\overline{MS}$ scheme, we have

$$Z_\phi = 1 + O\left(\lambda^2\right),\ Z_m = 1 + \frac{\lambda}{16\pi^2\epsilon} + O\left(\lambda^2\right),\ Z_\lambda = 1 + \frac{3\lambda}{16\pi^2\epsilon} + O\left(\lambda^2\right)$$

the the RG equation becomes

$$\left(1-\frac{3\lambda}{16\pi^2\epsilon}\right)\frac{d}{d\ln\mu}\left(\frac{3\lambda}{16\pi^2\epsilon}\right)+\frac{1}{\lambda}\frac{d}{d\ln\mu}\lambda+\epsilon=0$$

$$\left(\frac{3}{16\pi^2\epsilon}\lambda-\left(\frac{3}{16\pi^2\epsilon}\right)^2\lambda^2+1\right)\frac{1}{\lambda}\frac{d}{d\ln\mu}\lambda+\epsilon=0$$

drop the λ^2 term, we got

$$\left(\frac{3}{16\pi^2\epsilon}\lambda+1\right)\frac{1}{\lambda}\frac{d}{d\ln\mu}\lambda+\epsilon=0$$

$$\frac{1}{\lambda}\frac{d}{d\ln\mu}\lambda=-\epsilon\left(1-\frac{3}{16\pi^2\epsilon}\lambda\right)$$

$$\frac{d}{d\ln\mu}\lambda=\frac{3\lambda^2}{16\pi^2}-\lambda\epsilon$$

so the β-function of ϕ^4 theory with $\overline{MS}$ scheme is ($\epsilon\to 0$)

$$\beta(\lambda)=\frac{d}{d\ln\mu}\lambda=\frac{3\lambda^2}{16\pi^2}$$

Similarly, we can also get the RG evolution of mass via taking the derivative of μ on m_0

$$\ln m_0=\ln\left(Z_m^{1/2}Z_\phi^{-1/2}\right)+\ln m$$

$$0=\frac{1}{Z_m^{1/2}Z_\phi^{-1/2}}\frac{d}{d\ln\mu}\left(Z_m^{1/2}Z_\phi^{-1/2}\right)+\frac{1}{m}\frac{d}{d\ln\mu}m$$

substitute the renormalization parameters in to get

$$\left(1-\frac{\lambda}{16\pi^2\epsilon}\right)^{1/2}\frac{d}{d\ln\mu}\left(1+\frac{\lambda}{16\pi^2\epsilon}\right)^{1/2}+\frac{1}{m}\frac{d}{d\ln\mu}m=0$$

$$\frac{1}{2}\frac{1}{16\pi^2\epsilon}\left(1-\frac{\lambda}{16\pi^2\epsilon}\right)\frac{d}{d\ln\mu}\lambda+\frac{1}{m}\frac{d}{d\ln\mu}m=0$$

then we dropped λ^2 terms to get the anomalous dimension

$$\frac{1}{32\pi^2\epsilon}\left(\frac{3\lambda^2}{16\pi^2}-\lambda\epsilon\right)+\frac{1}{m}\frac{d}{d\ln\mu}m=0$$

$$\frac{1}{m}\frac{d}{d\ln\mu}m=\frac{\lambda}{32\pi^2}$$

<u>The Callan-Symanzik Equation (Peskin Ch.12.2)</u>

Except for coupling and mass, we can also consider the evolution of correlation functions, the bare correlation function can be written as

$$G_{n,0}(x_1,\ \dots x_n) = \langle\Omega|\mathcal{T}\{\phi_0(x_1)\dots\phi_0(x_n)\}|\Omega\rangle$$

since the vacuum state does not depend on renormalization, we have

$$G_{n,0}(x_1,\ \dots x_n) = Z_\phi^{n/2} G_n(x_1,\ \dots x_n)$$

do the same process as on coupling and mass above,

$$\ln G_{n,0} = \frac{n}{2}\ln Z_\phi + \ln G_n$$

$$0 = \frac{n}{2}\frac{d\ln Z_\phi}{d\ln\mu} + \frac{1}{G_n}\frac{d}{d\ln\mu}G_n$$

We know that the renormalized correlation function depends on $\lambda,\ m,\ \mu$, so we have

$$\frac{n}{2}\frac{d\ln Z_\phi}{d\ln\mu}G_n + \frac{d}{d\ln\mu}G_n = 0$$

$$\left(\frac{d\lambda}{d\ln\mu}\frac{\partial}{\partial\lambda} + \frac{dm}{d\ln\mu}\frac{\partial}{\partial m} + \frac{\partial}{\partial\ln\mu} + \frac{n}{2}\frac{d\ln Z_\phi}{d\ln\mu}\right)G_n = 0$$

we can define an anomalous dimension $\gamma_\phi = \dfrac{1}{2}\dfrac{d\ln Z_\phi}{d\ln\mu}$, then we got the Callan-Symanzik Equation

$$\left(\beta(\lambda)\frac{\partial}{\partial\lambda} + m\gamma_m\frac{\partial}{\partial m} + \frac{\partial}{\partial\ln\mu} + n\gamma_\phi\right)G_n(\lambda, m, \mu) = 0$$

<u>1-loop renormalization of QED</u>

The bare Lagrangian of QED is

$$\mathcal{L}_0 = i\overline{\psi}_0\not{\partial}\psi_0 - \frac{1}{4}F_0^2 - e\overline{\psi}_0\not{A}_0\psi_0 = iZ_2\overline{\psi}\not{\partial}\psi - \frac{1}{4}Z_3F^2 - eZ_1\overline{\psi}\not{A}\psi$$

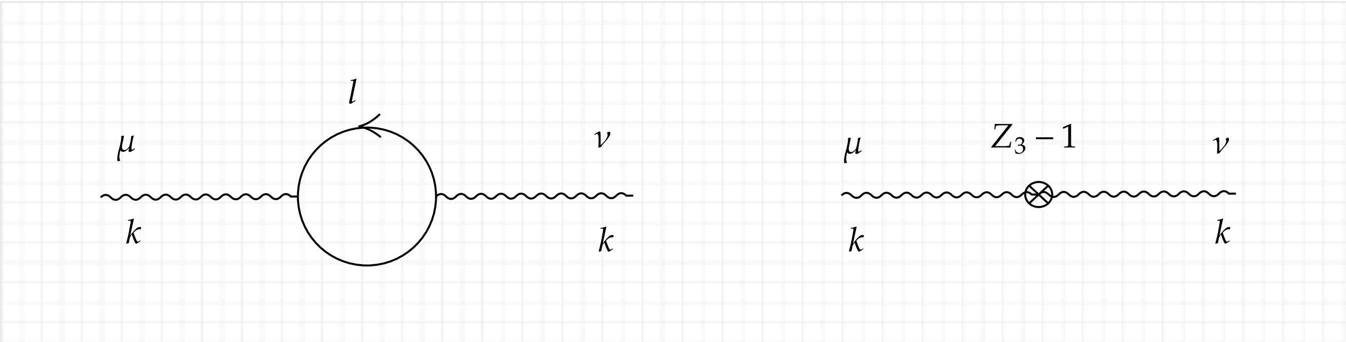
the free part and the perturbation can be separated as

$$\mathcal{L}_{\text{free}} = i\overline{\psi}\not{\partial}\psi - \frac{1}{4}F^2$$

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}' = \mathcal{L}_0 - eZ_1\overline{\psi}\not{A}\psi - \frac{1}{4}(Z_3-1)F^2 + i(Z_2-1)\overline{\psi}\not{\partial}\psi$$

where $\mathcal{L}'$ is the perturbation.

Consider the 1 loop correction of the photon self-energy, there are two diagrams



Then the self-energy can be written as

$$i\Pi^{\mu\nu}(k) = (-ie)^2(-1)\int \frac{d^4 l}{(2\pi)^4}\, Tr\left[\gamma^\mu \frac{i(\not{k}+\not{l})}{(k+l)^2+i\epsilon}\gamma^\nu \frac{i\not{l}}{l^2+i\epsilon}\right] - i(Z_3-1)\left(g^{\mu\nu}k^2 - k^\mu k^\nu\right)$$

the first term can be simplified as

$$-4e^2\int \frac{d^4 l}{(2\pi)^4}\frac{l^\mu(k+l)^\nu + l^\nu(k+l)^\mu - g^{\mu\nu}(l\cdot(k+l))}{l^2(k+l)^2}$$

use dimensional regularization and Feynman parameterization, we got

$$-4e^2\int \frac{d^d l}{(2\pi)^d}\int_0^1 dx\, \frac{l^\mu(k+l)^\nu + l^\nu(k+l)^\mu - g^{\mu\nu}(l\cdot(k+l))}{\left((1-x)l^2 + x(k+l)^2\right)^2}$$

replace the variable $q = l + xk$ and $\Delta = -x(1-x)k^2$, then we got

$$-4e^2\int \frac{d^d q}{(2\pi)^d}\int_0^1 dx\, \frac{(q-xk)^\mu(k+q-xk)^\nu + (q-xk)^\nu(k+q-xk)^\mu - g^{\mu\nu}((q-xk)\cdot(k+q-xk))}{\left(q^2-\Delta\right)^2}$$

drop those terms with odd power of q, then we got

$$-4e^2\int \frac{d^d q}{(2\pi)^d}\int_0^1 dx\, \frac{2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - g^{\mu\nu}\left(q^2 - x(1-x)k^2\right)}{\left(q^2-\Delta\right)^2}$$

using the identity (Peskin Eq.(A.41) - Eq.(A.45))

$$\int d^d q\, q^\mu q^\nu f\left(q^2\right) = \frac{g^{\mu\nu}}{d}\int d^d q\, q^2 f\left(q^2\right)$$

$$\int \frac{d^d q}{(2\pi)^d}\frac{1}{\left(q^2-\Delta\right)^2} = \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

$$\int \frac{d^d q}{(2\pi)^d}\frac{q^2}{\left(q^2-\Delta\right)^2} = \frac{-i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}}$$

it is simplified as

$$-4e^2\int \frac{d^d q}{(2\pi)^d}\int_0^1 dx\, \frac{1}{\left(q^2-\Delta\right)^2}\left[g^{\mu\nu}\left(\frac{2}{d}-1\right)q^2 + x(1-x)g^{\mu\nu}k^2 - 2x(1-x)k^\mu k^\nu\right]$$

$$= -4e^2\int_0^1 dx\left[g^{\mu\nu}\left(\frac{2}{d}-1\right)\frac{-i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}}+\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}x(1-x)\left(g^{\mu\nu}k^2-2k^\mu k^\nu\right)\right]$$

$$= -4e^2\frac{i}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\left[g^{\mu\nu}\left(-x(1-x)k^2\right)\frac{1}{1-d/2}+x(1-x)\left(g^{\mu\nu}k^2-2k^\mu k^\nu\right)\right]$$

$$= -8ie^2\left(g^{\mu\nu}k^2-k^\mu k^\nu\right)\int_0^1 dx\,x(1-x)\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}\Delta^{2-d/2}}$$

and we know that (Peskin Eq.(A.52))

$$\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}\Delta^{2-d/2}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\ln\Delta-\gamma+\ln(4\pi)+O(\epsilon)\right)$$

So, the self-energy can be written as

$$\Pi^{\mu\nu}(k) = \left(g^{\mu\nu}k^2-k^\mu k^\nu\right)\left[-8e^2\int_0^1 dx\,x(1-x)\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\ln\Delta-\gamma+\ln(4\pi)\right)-(Z_3-1)\right]$$
$$= \left(g^{\mu\nu}k^2-k^\mu k^\nu\right)\Pi\left(k^2\right)$$

in which

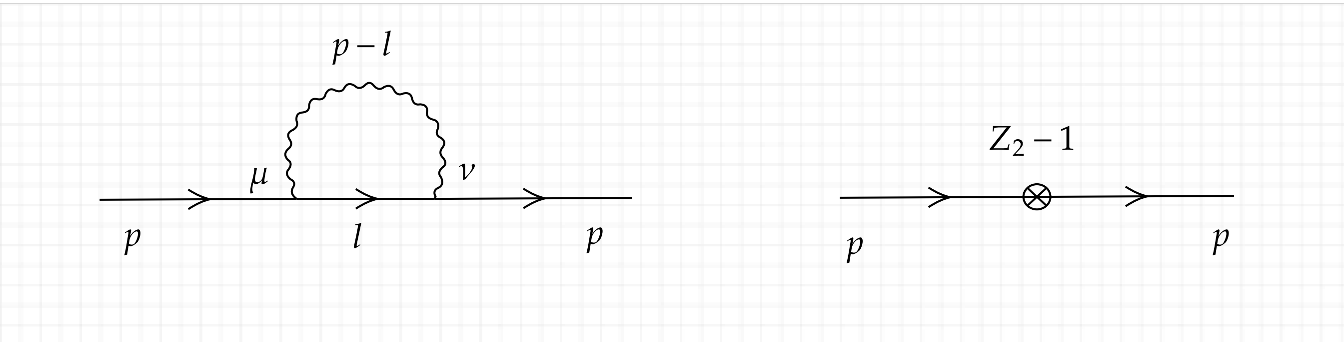
$$\Pi\left(k^2\right) = -\frac{e^2}{2\pi^2}\int_0^1 dx\,x(1-x)\left(\frac{2}{\epsilon}-\ln\Delta-\gamma+\ln(4\pi)\right)-(Z_3-1)$$

In $\overline{MS}$ scheme, the renormalization parameter is

$$Z_3 = 1-\frac{e^2}{2\pi^2}\int_0^1 dx\,x(1-x)\cdot\frac{2}{\epsilon} = 1-\frac{e^2}{6\pi^2}\frac{1}{\epsilon} = 1-\frac{2\alpha}{3\pi\epsilon}$$

where $\alpha = \dfrac{e^2}{4\pi}$.

Consider the 1 loop correction of the electron self-energy, there are two diagrams



Then the self-energy can be written as

$$i\Sigma(p) = (-ie)^2\int\frac{d^4l}{(2\pi)^4}\,\gamma^\mu\frac{i\not{l}}{l^2+i\epsilon}\gamma^\nu\frac{-ig_{\mu\nu}}{(p-l)^2+i\epsilon}+i(Z_2-1)\not{p}$$

The first term can be simplified as

$$-e^2\int\frac{d^4l}{(2\pi)^4}\frac{\gamma^\mu \not{l}\gamma_\mu}{(p-l)^2l^2} = -e^2\int\frac{d^4l}{(2\pi)^4}\frac{-2\not{l}}{(p-l)^2l^2}$$

here we used $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$.
Using Feynman parameter to get

$$-e^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dx\frac{-2\not{l}}{\left[x(p-l)^2+(1-x)l^2\right]^2}$$

replace variables by $q = l - xp$ and $\Delta = x(x-1)p^2$, using dimensional regularization and drop the odd power term of q, then we got

$$2e^2\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{x\not{p}}{\left(q^2-\Delta\right)^2}$$

using the identity (Peskin Eq.(A.41) - Eq.(A.45))

$$\int\frac{d^dq}{(2\pi)^d}\frac{1}{\left(q^2-\Delta\right)^2} = \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

we got

$$2e^2\int_0^1 dx\, x\not{p}\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

and we know that (Peskin Eq.(A.52)) in the $d = 4$ case,

$$\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}\Delta^{2-d/2}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\ln\Delta-\gamma+\ln(4\pi)+O(\epsilon)\right)$$

so the self-energy can be written as

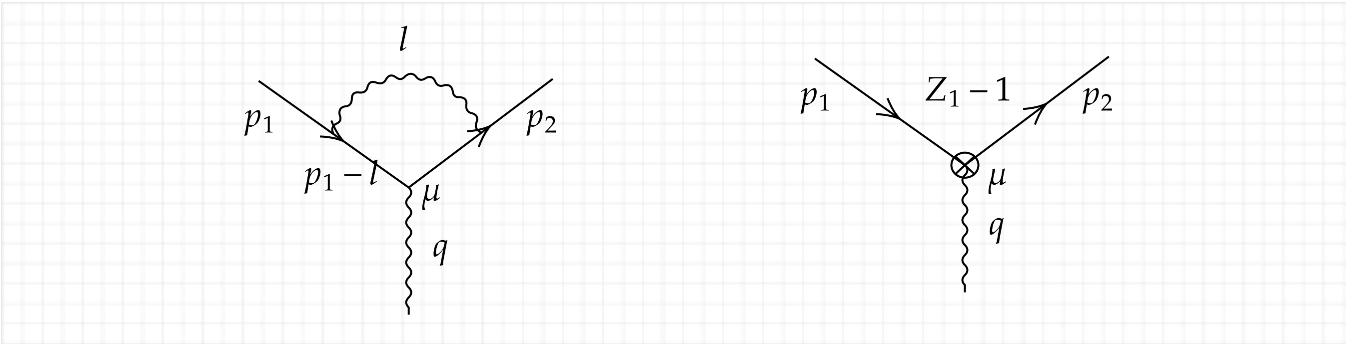
$$\begin{aligned}\Sigma(p) &= 2e^2\int_0^1 dx\, x\not{p}\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\ln\Delta-\gamma+\ln(4\pi)\right)+(Z_2-1)\not{p}\\ &= \not{p}\left[2e^2\int_0^1 dx\, x\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\ln\Delta-\gamma+\ln(4\pi)\right)+(Z_2-1)\right]\end{aligned}$$

In $\overline{MS}$ scheme, the renormalization parameter is

$$Z_2 = 1-\frac{2e^2}{(4\pi)^2}\int_0^1 dx\, x\cdot\frac{2}{\epsilon} = 1-\frac{e^2}{8\pi^2}\frac{1}{\epsilon} = 1-\frac{\alpha}{2\pi\epsilon}$$

which is consistent with Eq.(19.25) in Schwartz.

Consider the 1 loop correction of the electromagnetic vertex, there are two diagrams



the correction can be expressed as

$$-iV(p_1,p_2) = -i(Z_1-1)e\gamma^\mu + \int \frac{d^4 l}{(2\pi)^4}\left(-ie\gamma^\sigma\right)\frac{i\left(\not{p}_2-\not{l}\right)}{(p_2-l)^2}\left(-ie\gamma^\mu\right)\frac{i\left(\not{p}_1-\not{l}\right)}{(p_1-l)^2}\left(-ie\gamma^\nu\right)\frac{-ig_{\sigma\nu}}{l^2}$$

Similarly, dimensional regularization, then we got

$$-iV(p_1,p_2) = -i(Z_1-1)e\gamma^\mu - ie^3\int \frac{d^d l}{(2\pi)^d}\frac{-2\left(\not{p}_1-\not{l}\right)\gamma^\mu\left(\not{p}_2-\not{l}\right)}{(p_2-l)^2(p_1-l)^2 l^2}$$

Feynman parameterization

$$\frac{1}{(p_2-l)^2(p_1-l)^2 l^2} = 2\int_0^1 dx\int_0^{1-x} dy\,\frac{1}{\left[x(p_2-l)^2+y(p_1-l)^2+(1-x-y)l^2\right]^3}$$

then we simplify to get

$$-iV(p_1,p_2) = -i\frac{e^3}{8\pi^2}\left[\left(\frac{1}{\epsilon}-1+\frac{1}{2}\int dF_3\ln\left(\frac{\mu^2}{\Delta}\right)\right)\gamma^\mu+\frac{1}{4}\int dF_3\frac{N^\mu}{\Delta}\right]-ie(Z_1-1)\gamma^\mu$$

In $\overline{MS}$ scheme, the renormalization parameter is

$$Z_1 = 1-\frac{e^2}{8\pi^2}\frac{1}{\epsilon} = 1-\frac{\alpha}{2\pi\epsilon}$$

which is consistent with Eq.(19.56) in Schwartz.

In conclusion, take we got the 1-loop renormalization parameters of QED as

$$Z_3 = Z_A = 1-\frac{e^2}{6\pi^2}\frac{1}{\epsilon}$$

$$Z_2 = Z_\psi = 1-\frac{e^2}{8\pi^2}\frac{1}{\epsilon}$$

$$Z_1 = Z_e Z_\psi Z_A^{1/2} = 1-\frac{e^2}{8\pi^2}\frac{1}{\epsilon}$$

so the relation between the bare coupling and the renormalized coupling is

$$e_0 = Z_1 Z_2^{-1} Z_3^{-1/2} e\mu^{\epsilon/2}$$

$$\ln e_0 = \ln Z_1 - \ln Z_2 - \frac{1}{2}\ln Z_3 + \ln e + \frac{\epsilon}{2}\ln\mu$$

note here the power of μ is $\epsilon/2$, because $[A] = d/2 - 1$, $[\psi] = d/2 - 1/2$ and $[e] = 2 - d/2 = \epsilon/2$.
Take the derivative on $\ln\mu$, note $\ln Z_1 - \ln Z_2 = 0$, we got

$$
\begin{aligned}
-\frac{1}{2}\frac{d\ln Z_3}{d\ln\mu} + \frac{d\ln e}{d\ln\mu} + \frac{\epsilon}{2} = 0 \\
\frac{1}{2Z_3}\frac{d}{d\ln\mu}\left(\frac{e^2}{6\pi^2}\frac{1}{\epsilon}\right) + \frac{1}{e}\frac{de}{d\ln\mu} + \frac{\epsilon}{2} = 0 \\
\left(1 + \frac{1}{2}\frac{2e^2}{6\pi^2}\frac{1}{\epsilon}\right)\frac{1}{e}\frac{de}{d\ln\mu} = -\frac{\epsilon}{2} \\
\frac{de}{d\ln\mu} = -e\frac{\epsilon}{2}\left(1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon}\right) = \frac{e^3}{12\pi^2}
\end{aligned}
$$

so the β-function of QED with $\overline{MS}$ scheme is ($\epsilon \to 0$)

$$
\beta(e) = \frac{de}{d\ln\mu} = \frac{e^3}{12\pi^2}
$$

Another way to get the β-function

We can also use the Callan-Symanzik Equation to get the β-functon, check Peskin Ch.12.2, P411.

Comments

- Beta function depends on regularization and renormalization scheme, different scheme will leave different parameters to evolve;
- Beta functions from different scheme are consistent at the 1-loop level;
- If we want to compare with the scale evolution in experiment, we need to use on-shell scheme, but it is more complicated;
- We always have two ways to get the beta function, one is directly deal with the renormalization parameter of coupling, another is using the Callan-Symanzik Equation (need to calculate anomalous dimensions), the difference is just some more derivatives.