610 Midterm 1 review: Complex Analysis

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Cauchy-Rimann condition:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \text{ and } \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Cauchy's Theorem: If $f(z)$ is analytic in a simply connected bounded domain D, then for every simple closed path C in D we have

$$\oint_C f(z)\, dz = 0$$

[simply connected]: any loop can shrink to a point

Stoke's Theorem:

$$\oint \vec{A} \cdot d\vec{r} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$$

Cauchy's Integral Formula: If we know the boundary of an analytic function, we can know any point inside. If $f(z)$ is analytic in a simply connected domain D, then for any point $z = a$ in D and any closed path C in D which enclosed the point, we have

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$$

the integration being taken counter-clockwise.

Laurent Series: Expansion with negative power in the ring between singular points.

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots + b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \ldots$$

The coefficient b_1 of the term with power -1 is called the "principal" part.
With the equation

$$\oint \frac{dz}{(z-z_0)^n} = 2\pi i\, \delta_{n,1} \text{ , } n \in \mathcal{Z}$$

we have $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}$ and $b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$.

The Residue Theorem: [show all steps]

$$\oint_C f(z)dz = 2\pi i \cdot \sum \left\{\text{Res}[z_i]\right\}$$

e.g.

$$f(z) = \frac{z-2}{\left(z^2-4\right)(z-1)^3}$$

$z_1 = 1$ is a pole of order 3, $z_2 = -2$ is a pole of order 1, $z_3 = 2$ is not a pole.

For the pole of order 1: Res$[z_i] = \lim_{z \to z_0} (z - z_0) f(z)$.

For the pole of order n: Res$[z_i] = \dfrac{1}{(n-1)!} \lim_{z \to z_0} \left(\left[(z - z_0)^n f(z) \right]^{(n-1)} \right)$, here $n - 1$ means the order of derivative.

Branch cut

Principal Value

Mapping between z-plane and ω-plane

e.g.

The mapping $\omega = \dfrac{1}{z}$ maps every circle or straight line onto a circle or straight line.

proof:

$$z\bar{z} = x^2 + y^2$$
$$x = \frac{1}{2}(z + \bar{z})$$
$$y = \frac{1}{2i}(z - \bar{z})$$

Liouville's Theorem: If $f(z)$ is analytic and bounded in absolute value in the entire complex plane, then it must be a constant.

The method of Steepest Descent: [figure with contour]

Consider integral $I(s) = \int_C g(z) e^{sf(z)} dz$ where $s \gg 1$. Find z_0 that is a saddle point of $f(z)$, $f''(z_0) = |f''(z_0)| e^{i\theta}$.

Then, $I(s) = \dfrac{\sqrt{2\pi} \cdot g(z_0) \cdot e^{sf(z_0)} e^{i\alpha}}{\sqrt{s |f''(z_0)|}}$, where $\alpha = -\dfrac{\theta}{2} \pm \dfrac{\pi}{2}$.

Identity Theorem: $f_1(z)$ and $f_2(z)$ are analytic within region D. If these 2 functions coincide in the neighbourhood of a point z_0 in D or on the segment of a curve lying in the D, then they coincide in D.

Analytic Continuation: [figure with analytic region and branch cut]

If you have an analytic function on x-axis, it can be expanded to a complex form in only one way to make it analytic.

e.g.

$$f_1(z) = 1 + z + z^2 + \ldots, \ |z| < 1$$

$$f_2(z) = \frac{2}{3} + \left(\frac{2}{3}\right)^2\left(z+\frac{1}{2}\right) + \left(\frac{2}{3}\right)^3\left(z+\frac{1}{2}\right)^2 + \ldots,\ |z+\frac{1}{2}| < 1$$

Two forms represent the same analytic function $f(z) = \dfrac{1}{1-z}$ in the complex plane.

e.g.

$$I\left(a^2\right) = \int_{-\infty}^{\infty} \frac{dx}{x^2 - a^2 + i\epsilon},\ \epsilon \to 0$$

a) Evaluate for $a^2 > 0$.
b) Evaluate for $a^2 < 0$.
c) By promoting a^2 to a complex number and analytically continuing $I\left(a^2\right)$ to verify a) and b) are consistent.

$I\left(a^2\right)$ is singular at $z_1 = a - i\hat{\epsilon}$ and $z_2 = -a + i\hat{\epsilon}$ where $\hat{\epsilon} = \dfrac{\epsilon}{2a}$.